Journal of the Nigerian Association of Mathematical Physics
***Volume* 24 (July, 2013), pp 85 – 88**

On Uniform Convergence to a Solution of an Initial Value Problem of an Nth Order Linear Differential System

Eze E.O., Ogbu H. M. and Aja R.O.

Department Of Mathematics, Michael Okpara University of Agriculture, Umudike-Umuahia, Abia state Nigeria

Abstract

Uniform convergence remains the "best" method for approximation of solutions of an initial value problem of an nth-order differential system out of all types of convergences. Our objective in this paper is to approximate a solution to an initial value problem, by constructing a sequence of functions that will converge uniformly to a solution. The proof was shown to exist from uniform Cauchy criterion and further confirmed by Lebesgue Dominated Convergence Theorem as against the method of Induction that is popularly used. The results showed that even Picard's theorem is a direct consequence of uniform convergence which implies all other types of convergences but the converse is not true. Again both continuity and boundedness are direct consequences of the proof of Weierstrass theorem and depend on the uniform Cauchy criterion and hence, the solution to our problem.

Keywords: Uniform convergence; Uniform Cauchy Criterion, Lebesgue dominated convergence theorem, nth –order differential system, Sequence of functions.

1.0 Introduction

Uniform convergence being one of the properties of solutions of Differential systems has largely remained one of the "best" methods of approximations of solutions, although very difficult to construct, it has most useful consequences.

There are many ways to prove the existence of a solution to an ordinary differential equation. The simplest is to find one explicitly. This is a good approach for separable or exact equations or linear equations with constant coefficients but unfortunately, there are many equations that cannot be solved by elementary methods, so attempting to prove the existence with this approach may not be visible.

An alternative approach is to approximate a solution to an initial value problem by constructing a sequence of functions that converge uniformly to a solution. This approach is due to Picard [1]. Thus, uniform convergence is the best kind of convergence. It has most useful consequences but it is also difficult to achieve. In most cases we settle for pointwise convergence or L^2-convergence instead [2]. Though uniform convergence implies all other types of convergences but the converse is not true. Also, continuity and boundedness are direct consequences of uniform convergence; and even Picard's theorem. Again, even the proof of Weierstrass theorem depends heavily on the uniform Cauchy criterion although the former is on uniform absolute convergence and the later is for uniform and non-uniform convergence.

In this paper our objective is to construct a sequence of functions and then prove that it converges uniformly to a solution of an initial value problem of an nth-order linear differential system of the format

Corresponding author: ***Eze E.O.,*** E-mail: obinwanne_eze@yahoo.com, Tel.: +2348033254972

$$\dot{X} = F(t,x) \quad (1.1)$$

With initial condition:

$$\left.\begin{aligned} x(to) &= xo \\ x(t_1) &= x_1 \\ x(t_2) &= x_2 \\ x(t_3) &= x_3 \\ . \quad & \quad . \\ . \quad & \quad . \\ . \quad & \quad . \\ x(t_n) &= x_n \end{aligned}\right\} \quad (1.2)$$

We now employ Picard's iteration on the constructed sequence of functions which is defined recursively as follows:

$$x(t_o) = x_o$$

$$\left.\begin{aligned} x_1(t_1) &= x_o + \int_{t_o}^{t} F(S, x_0(s))ds \\ x_2(t_2) &= xo + \int_{t_o}^{t} F(S, x_1(s))ds \\ . \quad & \quad . \\ . \quad & \quad . \\ . \quad & \quad . \\ x_n(t_n) &= x_o + \int_{t_o}^{t} F(S, x_{n-1}(s))ds \end{aligned}\right\} \quad (1.3)$$

If this sequence of functions converge uniformly to a function $x(t)$, then this function is the solution of (1.1) and (1.2) [1,3].

2.0 Some Theorems And Definitions Underlying Our Concept

We now state some theorems and definitions which will be an asset for our proof.

Theorem 2.1: (Weierstrass M-Test)

Assume $\{X_N\}_{N=1}$ is a sequence of functions defined in an open interval $a < t < b$. suppose that $\{M_N\}_{N=1}$ is a sequence of positive constants such that $|X_N(t)| \leq M$ for all $a < t < b$.

If is convergent, then converges uniformly for all $a < t < b$

Proof: see p. 73 of [1]

<u>Theorem 2.2</u> [Lebesgue Dominated Convergence Theorem]

Let $\{f_n(x)\}$ be a sequence of functions measurable on E such that

$\lim_{n\to\infty} f_n(x) = f(x)$

Then, if there exists a function $M(x)$ integrable on E such that:

$|f_n(x)| \leq M$ for all n.

We have:

lim =

n→∞

= see p. 74 of 4

Proof: See p.84 of [4]

The very notion of Existence and Uniqueness of solutions of any initial value problem of differential systems lie in Picard's theorem and we re-state as follows:

Theorem 2.3(Picard)

Suppose the functions $f(t,x)$ and $\frac{\partial f}{\partial x}$ are continuous for all (t,x) in the rectangular plane R and bounded, that is;

(i) $|f| \leq K$

(ii) $\left|\frac{df}{dx}\right| \leq M$

For all (t,x) in R, then our initial value problem of (1.1) and (1.2) has at most one solution $x(t)$. Hence, it has precisely one solution.

Proof: See pp 93-94 of [3] and also [1]

Theorem 2.4 [Uniform Cauchy Criterion]

Let $\{f_n\}$ be a sequence of functions defined on a subset D of R. then, there exists a function f such that $\{f_n\}$converges uniformly to f on D if and only if the following condition is satisfied:

given □ > 0 there exists N=$N(t)$ such that:

$|f_n(x) - f_m(x)| < \in$ for every x □D and for every $n, m \geq N$.

Proof: See p. 37 0f [5]

Definition 2.5(a)

Let $\{g_1, g_2, g_{3,\ldots}\}$ be function from $X \to R$and $f: X \to R$ be some other function. The sequence $\{g_1, g_2, g_{3,\ldots}\}$ converges uniformly to f if $\lim_{n\to\infty} \|g_n - f\|_\infty = 0$ see p 127 of [2]

Definition 2.5(b)

Let $\{f_n\}$be a sequence of function defined in a subset D of R. then $\{f_n\}$ is said to converge uniformly on D if:

given $\in > 0$ there exists $N \in N$ such that :

$\left| f_n(x) - f(x) \right| < \in$ for every $x \in D$

See p.36 of [5] and p 127 of [2]

3.0 Methodology

OGBU and Eze and Finan M.B have proved uniform convergence using the method of induction but our approach will be different in that we will offer the proof by using theorem 2.4[Uniform Cauchy Criterion] and further augmentation of the results by theorem (2.2) to show that our constructed sequence of functions (1.3) converges uniformly to a solution $x(t)$ of (1.1) and (1.2)

4.0 Main Results

Invoking our theorem (2.4) and applying it to (1.3), we obtain the following

Proof (□) Suppose $x_n(t)$ □$x(t)$ uniformly on the interval I = $[to - h;\ to + h]$.

Then given ∈ > 0, we choose our $N = N(\in)$ such that:

$$|x_m(t) - x(t)| < \frac{\in}{2} \tag{4.1}$$

For all $n > N(t)$ and for all $t \in I$.

We have;

$$|x_m(t) - x_n(t)| \square |x_m(t) - x(t)||x(t) - x_n(t)|$$
$$< \frac{\in}{2} + \frac{\in}{2}$$
$$= \in \tag{4.2}$$

⟸ On the other hand, suppose $x_n(t)$ satisfies uniform Cauchy Criterion. Then for each fixed t □I the numerical sequence $x_n(t)$ satisfies the ordinary Cauchy criterion. So there exists a limit which we call $x(t)$ defined on

I = $[to - h;\ to + h]$ such that:

$x_n(t) \to x(t)$ pointwise on I

Remark: Note that this pointwise convergence on I implies Uniform Convergence on I but the converse is false.

We now show that $x_n(t) \to x(t)$ uniformly on I. Infact, for any given $\epsilon > 0$ we can choose $N = N(t)$ such that :

$$|x_m(t) - x_n(t)| < \in \tag{4.3}$$

For all $m, n > N$ and all $t \in I$.

Furthermore, fix $n > N$ and t □I and let $m \to \infty$, we obtain;

$|x(t) - x_m(t)| \leq \in$ for all $t \in I$

Remark: The choice of N depends only on ∈ and not on ∈ and $x \in D$ as is the case in pointwise convergence.

5.0 Further Results

To further confirm our above assertion, we employ 2.2 on Lebesgue Dominated Convergence Theorem and apply as thus;

$$\lim_{n\to\infty} \|x_n(t) - x(t)\|_\infty \|x_n(t) - x(t)\|_\infty = 0$$

$$\Longrightarrow \lim\ \sup_{n\to\infty\ t\in I} |x_n(t) - x(t)| = 0 \tag{5.1}$$

$$\lim_{n\to\infty} x_n(t) = x(t) \tag{5.2}$$

We know from (1.3) that :

$$x_n(t) = \int_{t_o}^{t} F\big(S, x_{n-1}(s)\big) ds \tag{5.3}$$

We write (5.2) as

$x(t) = \lim_{n\to\infty} x_n(t)$

Therefore,

$$x(t) = \lim_{n\to\infty}\left[x_0 + \int_{t_o}^{t} F\big(S, x_{n-1}(s)\big)ds\right] \quad (5.4)$$

$$= x_0 + \left[\lim_{n\to\infty}\int_{t_o}^{t} F\big(S, x_{n-1}(s)\big)ds\right]$$

$$= x_0 + \left[\int_{t_o}^{t} F\big(S, x_{n-1}(s)\big)ds\right]$$

$$= x_0 + \int_{t_o}^{t} F\big(S, x_{n-1}(s)\big)ds\ldots \quad (5.5)$$

This shows that $x(t)$ is a solution to our integral equation of (1.3) and therefore a solution.

Discussion

1. Uniform Convergence Implies Pointwise Convergence implies convergence in L^2-space but the converse is not true(Note the later, the sequence is uniformly bounded and X is compact) [2]
2. Uniform convergence implies semi-uniform convergence implies pointwise convergence but however the converse is false[2]
3. Hence to understand convergence in general, it is sufficient to understand uniform convergence to the constant of zero function.
4. We can deduce that both boundedness and continuity can be direct consequences of uniform convergence and lends a confirmatory hand to Picard's theorem on existence and Uniqueness of solutions.
5. A sequence $\{x_k\}$ in R converges to a point in R if and only if $\lim_{n\to\infty} sup\{x_k\} = \lim_{k\to\infty} inf\{x_k\} = x$ [6]
   Note: This point x is a unique point which coincides with a unique solution but we know that uniqueness of a solution implies the existence of a solution but the converse is not true.
6. A sequence $\{x_k\}$ in R converges to a point x in R if and only if every subsequence of $\{x_k\}$ converges to x. This point is a unique point which coincides with a unique solution. (see pp 213 – 218 of [6])

Conclusion

The conclusion shows that x(t) is a solution to our integral equation (1.3) and therefore a solution to our initial value problem of (1.1) and (1.2) but the deductions from the proof are similar to the results obtained in [6].

References

[1] Finan M.B(1999). A First Course in elementary differential equations; Arkansas technical University press.

[2] Pivato Marcus(2007). Linear partial differential equations and Fourier theory, Tvente University Press, Peterborough, Ontario, Canada.

[3] Ogbu H.M and Eze; E.O (2012). An insight on existence and uniqueness of solutions for an initial value problem of a first order differential equation. International Research and Development Institute vol. 9, No. 5, pp 94-97

[4] Murray R. Spiegel [2009] Schaum's Outline of Theory and Problems of Real Variables: Lebesgue Measure and Integration With Applications to Fourier Series. The McGraw-Hill Companies Inc. USA.

[5] Chidume C.E and Chidume C.O(2003); Foundations of Riemann Integration "Abdussalam International Centre for Theoretical Physics-Trieste Italy

[6] Eze, E.O, Ugbene I.J and Ogbu, H.M(2012) On the demand for uniqueness of solutions of an initial value problem of a first order linear differential system. " Icastor Journal of Mathematical Sciences, India Vol. 6, No.2, pp 213-218

[7] Rangarajan K.Sundaram[1999]; A first Course in Optimization theory; Cambridge University Press.