Journal of the Nigerian Association of Mathematical Physics
Volume **24 (July, 2013), pp 79 – 84**
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A Hybrid Linear Collocation Multistep Method for Solving Initial Value Problems of First Order Ordinary Differential Equations

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Abstract

Our focus in this paper is the proposition of six – step, hybrid linear multistep method with three off – step points for the numerical solution of initial value problems of first order ordinary differential equations. The technique of interpolation and collocation was used to derive a continuous scheme, from where the main method and additional schemes were obtained. The schemes were then applied in block form as simultaneous integrators over non – overlapping intervals on initial value problems of first order ordinary differential equations. The basic properties of the method were analyzed and the results showed that the method is consistent, zero – stable, convergence and accurate.

Keywords: Hybrid, block method, linear multi step, numerical, interpolation and collocation.

1.0 Introduction

Ordinary differential equations often arise from many processes in the fields of sciences, Management and Engineering where the rates of change of one or more quantities with respect to one independent variable occur. Wide varieties of natural phenomena in the aforementioned fields are modeled by ordinary differential equations of the general form:

$$y' = f(x,y), \qquad y(x_0) = y_0, x \in [a,b], \tag{1}$$

Where y' indicates the derivative of dependent variable y with respect to x and function f satisfies the Lipschitz condition of the existence and uniqueness of solution to the ordinary differential equation.

We seek a solution to (1) in the range of: $[a \le x \le b]$ where a and b are finite and the problem has a unique continuously differentiable solution. We consider a sequence of points: $\left\{ x_n / x_n = a + nh, n = 0,1,2 \ldots \frac{b-a}{h} \right\}$ where the parameter h which is a constant is the step length. It should be noted that the majority of computational methods for the solution of (1) is of property called discretization, meaning that we seek an approximate solution to the problem, not on continuous interval $[a \le x \le b]$, but on the discrete point set $\{x_n\}$.

The K – step Linear Multistep method (LMM) for the solution of (1) is generally written as:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\left[\sum_{j=0}^{k} \beta_j f_{n+j} \right] \tag{2}$$

Which has 2k+1 unknown $\alpha's$ and $\beta's$ and therefore can be of order 2k. Dalquist [1] postulated that the order of the Linear Multistep Method (2) cannot exceed k+1 or k+2 when k is odd or even, respectively, for the method to be stable. To overcome this postulated barrier, many early researchers in Numerical computing of ordinary differential equations such as [2], [3], and [4] proposed the modified forms of (2). Their works lead to what is christened generally as hybrid methods usually obtained by incorporating off – step points to (2) leading to:

$$\sum_{j=1}^{k} \alpha_j y_{n+j} = h\left[\sum_{j=0}^{k} \alpha_j f_{n+j} + \beta_v f_{n+v} \right], \tag{3}$$

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Where v which is usually in the interval $[0,1]$ is called hybrid points. [5] observed that deriving this kind of methods is more tedious due to the occurrence of the fractional off – step points which increases the number of predictors needed to implement the methods. To over step this barrier, block method of implementation of Linear Multistep Methods which is usually self starting is now being commonly adopted by researchers. The works of [6] - [8] for the solution of (1), attest to the fact that proposition of hybrid continuous collocation methods is now commonly in vogue for the numerical solution of ordinary differential equations, apparently because they are efficient, accurate and adequate.

The solution of (1) has been equally discussed extensively by various researchers [9 – 15].

Collocation methods for generating computational methods of the form (1) or its modified form (2) has its origin dated back as far as 1965 when Lanczos [16] introduced the standard collocation method with some selected points for the numerical integration of ordinary differential equations. However, it should be noted that earlier researchers that proposed collocation methods for solving ODEs developed discrete schemes; it was Ortiz [18] that improved on earlier works in this area and showed that traditional multistep methods including the hybrid ones can be made continuous through the idea of multistep collocation scheme against the discrete schemes, since global error estimates can be attained in addition to better approximation at all interior points. Therefore, the major advantage of this aforementioned innovation is that the introduction of continuous collocation methods has bridged the gap between the discrete collocation method and the conventional multistep method.

Consequently, in this paper, we propose six – step hybrid linear multistep method with three off – step points by employing multistep collocation approach which produces a class of nine schemes of order of accuracy eleven for the numerical integration of initial value problems of first order ordinary differential equations.

2.0 Derivation of the Method.

We assume an approximate solution to equation (1) to be a continuous solution of the form:

$$y(x)=\sum_{j=0}^{p+q-1}\alpha_j x^j \qquad (4)$$

Such that $x\in[a,b]$, where a_j are unknown coefficients of the polynomial basis function of degree $p+q-1$, where the number of interpolation points p and the number of collocation points q are respectively chosen to satisfy $1\le p\le k$ and $q\succ 0.$ Note that the step number of the method is represented by $k\succ 1.$

We seek a K – step multistep collocation method of the form:

$$\sum_{j=1}^{6}\alpha_j y_{n+j}=h\left[\sum_{j=0}^{6}\alpha_j f_{n+j}+\beta_v f_{n+v}\right], \qquad (5)$$

Where α_j and β_j are coefficients and $v=\left\{21/4,11/2,23/4\right\}$ are hybrid points.

We construct a k – step continuous hybrid multistep method with $x^j,\ j=0,1,\ldots,10,\ p=1,q=10,k=6$ by imposing the above condition, we have:

$$\sum_{j=0}^{10} ja_j x_{n+1}^{j-1}=f_{n+i},\, i=\left\{0,1,2,3,4,5,21/4,11/2,23/4,6\right\} \qquad (6)$$

$$\sum_{j=0}^{10} ja_j x_{n+1}^{j-1}=y_{n+i},\, i=5 \qquad (7)$$

Where n in (6) and (7) above is the grid index.

From equations (6) and (7) we obtain a system of $p+q$ equations which is solved to obtain the coefficients a_j's by Gaussian elimination method. By putting the values of these coefficients a_j's so obtained into equation (4), we obtain the six – step continuous hybrid method. On evaluating the continuous scheme at points:

$x=\left\{x_n,x_{n+1},x_{n+2},x_{n+3},x_{n+4},x_{n+5},x_{n+21/4},x_{n+11/2},x_{n+23/4},6\right\}$, we obtain the following nine discrete schemes:

$$y_{n+6}-y_{n+5}=h\begin{bmatrix}\frac{17}{57380400}f_n-\frac{1697}{43953800}f_{n+1}+\frac{149}{5896800}f_{n+2}-\frac{179}{1496880}f_{n+3}+\\ \frac{8}{14175}f_{n+4}+\frac{30407}{453600}f_{n+5}+\frac{514048}{1342575}f_{n+21/4}+\\ \frac{9872}{93555}f_{n+11/2}+\frac{25126912}{68139225}f_{n+23/4}+\frac{3797}{50400}f_{n+6}\end{bmatrix} \quad (8)$$

$$y_{n+23/4}-y_{n+5}=h\begin{bmatrix}-\frac{33993}{185702809}f_n+\frac{3081}{1247805440}f_{n+1}-\frac{23293}{1363145800}f_{n+2}+\frac{181581}{201850880}f_{n+3}-\\ \frac{416763}{734003200}f_{n+4}+\frac{36947173}{367001600}f_{n+5}+\frac{337479}{1237600}f_{n+21/4}+\\ \frac{2184009}{7884800}f_{n+11/2}+\frac{147283}{1416800}f_{n+23/4}-\frac{559793}{146800}f_{n+6}\end{bmatrix} \quad (9)$$

$$y_{n+11/2}-y_{n+5}=h\begin{bmatrix}-\frac{2813}{5875752960\ 0}f_n+\frac{75041}{1125218304\ 00}f_{n+1}-\frac{14647}{3019161600}f_{n+2}+\\ \frac{53419}{1916006400}f_{n+3}-\frac{9931}{46448640}f_{n+4}+\frac{109198753}{116121600}f_{n+5}+\\ \frac{607576}{1879605}f_{n+21/4}+\frac{97553}{1069200}f_{n+11/2}-\frac{20344}{9734175}f_{n+23/4}-\frac{877}{25804800}f_{n+6}\end{bmatrix}$$

(10)

$$y_{n+21/4}-y_{n+5}=h\begin{bmatrix}-\frac{188267}{1367447961\ 600}f_n+\frac{54062101}{2880558858\ 2400}f_{n+1}-\frac{2040799}{1545810739\ 20}f_{n+2}+\\ \frac{35080343}{4904976384\ 00}f_{n+3}-\frac{28570219}{5945425920\ 0}f_{n+4}+\frac{3025808117}{2972712960\ 0}f_{n+5}+\\ \frac{7839803}{42962400}f_{n+21/4}-\frac{7662463}{174182400}f_{n+11/2}+\frac{5040829}{436091040}f_{n+23/4}-\frac{9342617}{6606028800}f_{n+6}\end{bmatrix} \quad (11)$$

$$y_{n+4}-y_{n+5}=h\begin{bmatrix}-\frac{1523}{5738400}f_n+\frac{165791}{439538400}f_{n+1}-\frac{37}{12960}f_{n+2}+\frac{1365813}{7484440}f_{n+3}-\\ \frac{19919}{56700}f_{n+4}+\frac{1148153}{453600}f_{n+5}+\frac{2656256}{722925}f_{n+21/4}-\\ \frac{1212208}{46775}f_{n+11/2}+\frac{12470272}{13627845}f_{n+23/4}-\frac{6611}{50400}f_{n+6}\end{bmatrix} \quad (12)$$

$$y_{n+3}-y_{n+5}=h\begin{bmatrix}\frac{1847}{14345100}f_n-\frac{55829}{27471150}f_{n+1}+\frac{14143}{737100}f_{n+2}-\frac{180286}{467775}f_{n+3}-\\ \frac{15011}{11340}f_{n+4}+\frac{33293}{28350}f_{n+5}-\frac{7143424}{1879605}f_{n+21/4}+\\ \frac{1670656}{467775}f_{n+11/2}-\frac{101777408}{68139225}f_{n+23/4}+\frac{1513}{6300}f_{n+6}\end{bmatrix} \quad (13)$$

$$y_{n+2}-y_{n+5}=h\begin{bmatrix}\frac{-447}{708400}f_n+\frac{449}{361760}f_{n+1}-\frac{26683}{72800}f_{n+2}-\frac{41029}{30800}f_{n+3}-\\ \frac{72}{175}f_{n+4}-\frac{36457}{5600}f_{n+5}+\frac{489472}{38675}f_{n+21/4}-\\ \frac{2928}{275}f_{n+11/2}+\frac{514048}{1201755}f_{n+23/4}-\frac{759}{1120}f_{n+6}\end{bmatrix} \quad (14)$$

$$y_{n+1} - y_{n+5} = h\left[\begin{array}{l} \frac{18712}{3586275} f_n - \frac{238334}{722925} f_{n+1} - \frac{20536}{14175} f_{n+2} - \frac{19144}{93555} f_{n+3} - \\ \frac{39992}{14175} f_{n+4} + \frac{251126}{14175} f_{n+5} - \frac{4194304}{103275} f_{n+21/4} + \\ \frac{3407872}{93555} f_{n+11/2} - \frac{54525952}{3586275} f_{n+23/4} + \frac{3896}{1575} f_{n+6} \end{array}\right] \qquad (15)$$

$$y_{n} - y_{n+5} = h\left[\begin{array}{l} \frac{-619915}{2295216} f_n - \frac{31097225}{17581536} f_{n+1} + \frac{16025}{18144} f_{n+2} - \frac{1626775}{299376} f_{n+3} + \\ \frac{23825}{2268} f_{n+4} - \frac{2325985}{18144} f_{n+5} + \frac{1177600}{4131} f_{n+21/4} - \\ \frac{4775600}{18711} f_{n+11/2} + \frac{291441600}{2725569} f_{n+23/4} - \frac{35275}{2015} f_{n+6} \end{array}\right] \qquad (16)$$

These nine discrete schemes (8) – (16) will be arranged in block form as simultaneous integrator of test problems of initial value first order ordinary differential equations.

3.0 Analysis of the Basic Properties of the Method.

3.1 Order of accuracy and Error Constant.

In line with [17], the local truncation error associated with K – step linear multistep method (2), is taken to be linear difference operator:

$$L[y(x),h] = \sum_{j=0}^{k}\{\alpha_j y(x_{n+j}) - h\beta_j y(x_{n+j})\} \qquad (17)$$

Equation (17) can be expanded as a Taylor's series about the point x if $y(x)$ is sufficiently differentiable to obtain the expression:

$$L[y(x),h] = C_0 y(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) + \ldots + C_2 h^q y^q(x_n) + \ldots, \qquad (18)$$

where $C_{q,}\ q = 0,1,\ldots,$ are the constant coefficients given as:

$$C_0 = \sum_{j=0}^{k}\alpha_j, \qquad (19)$$

$$C_1 = \sum_{j=0}^{k} j\alpha_j, \qquad (20)$$

And $$C_q = \frac{1}{q!}\left[\sum_{j=0}^{k} j\alpha_j - q(q-1)\left(\sum_{j=0}^{k} j^{q-1}\beta_j + \sum_{j=0}^{k} v^{q-1}\beta_{vj}\right)\right], \qquad (21)$$

According to (18), in line with [17], we say that the hybrid k – step, linear multistep method (5) has order p if $C_0 = C_1 = \ldots = C_{p-1} = C_p$ and $C_{p+1} \neq 0$. Thus C_{p+1} is the error constant of the method. From our calculation, subjecting our schemes to equations (18)-(21), it is established that our hybrid linear multistep schemes have high order of accuracy $P = (11 \quad 11 \quad 11 \quad 11 \quad 11 \quad 11 \quad 11 \quad 11 \quad 11)^T$ and relatively small error constants.

3.2 Consistency

A linear multistep method (5) is said to be consistent if:

(i) The order $p \geq 1$, (ii) $\sum_{j=0}^{k}\alpha_j = 0,$

(ii) $\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k}\beta_j,$ (iv) $\rho(1) = 0, \rho'(1) = \sigma(1),$

Where ρ and σ are the first and second characteristic polynomials of equation (5), the general form of our method. On applying these aforelisted definitions to our schemes (8-16), they were found to be consistent.

3.3 Zero Stability of the method.

A linear multistep method of the form (5) is said to be Zero stable if no roots of the first characteristic polynomial $\rho(r)$ has modulus greater than one, and if every root of the modulus one is simple [17]. In the same way, by applying this definition to our schemes (8-16), they were found to be Zero stable.

4.0 Implementation of the method.

Our derived schemes are implemented by combining (8)-(16) together as simultaneous integrator for the initial value problems of first order ordinary differential equations without requiring starting values and predictors. In doing this, we proceed by explicitly obtaining initial conditions at $x_{n+6}, n = 0,6,..., N-6,$ using the computed values: $y(x_{n+6}) = y_{n+6}$ over sub intervals $[x_0, x_6],...,[x_{N-6}, x_N]$, specifically, we use equations (8-16) by setting n = 0, $\mu = 0.$ we obtain simultaneously $\left(y_1, y_2, y_3, y_4, y_5, y_{21/4}, y_{11/2}, y_{23/4}, y_6\right)^T$, over the sub interval, $[x_0, x_6]$, since y_0 is known from the initial value problem (1).

In the same way, by setting n = 6, $\mu = 1$, we obtain simultaneously:

$\left(y_7, y_8, y_9, y_{10}, y_{11}, y_{45/4}, y_{23/2}, y_{47/4}, y_{12}\right)^T$, Over the sub interval: $[x_6, x_{12}]$ as y_6 is known from the previous block, T being the transpose. We then continue this process until we wish to stop the iterations. Hence, the sub – interval do not overlap, thus, the solution obtained from here are more accurate than those obtain in the conventional fashion. However, linear problems are solved from the start with Gaussian elimination method using partial pivoting, while we apply modified Newton - Raphson method for nonlinear problems.

4.1 Numerical Results.

Using some test problems, we illustrate the numerical schemes (8-16) to test the suitability and performance of the method. All calculations and computer program are carried out with the aid of MATLAB software. The results are presented in tabular form in table 1 and 2 shown below.

Test Problem 1.

We consider an initial value problem:

$$y' = -20y + 20\sin x + \cos x \qquad y(0) = 1 \text{ with h = 0.01}$$

Whose exact solution is $y(x) = e^{-20x} + \sin x$

The results are as shown in Table 1.

Table 1. Results for Problem 1.

x	Exact Solution	Numerical Solution	Error.
0	1.000000000000	1.000000000000	1.0000
0.1	0.235146948000	0.235146947998	2.0624×10^{-12}
0.2	0.216984969683	0.216984969672	1.1206×10^{-11}
0.3	0.297998958838	0.297998958826	1.2421×10^{-11}
0.4	0.389753804936	0.389753804913	2.2462×10^{-11}
0.5	0.479470938533	0.479470938511	2.1026×10^{-11}
0.6	0.564648617607	0.564648617606	1.2262×10^{-11}
0.7	0.644218517832	0.644218517811	2.1610×10^{-11}
0.8	0.717356203290	0.717356203270	2.0416×10^{-11}
0.9	0.841470986866	0.841470986834	3.2267×10^{-11}

Test Problem 2.

We consider the initial value problem given by:

$$y' = 8(x - y) + 1, \qquad y(0) = 2, with\, h = 0.01$$

Whose exact solution is given by $y(x) = x + 2e^{-8x}$

Our results were compared with that of [6]. The results are as shown in Table 2.

Table 2. Results for Problem 2.

x	Exact Solution	Numerical Solution	Error.	Error in [6]
0	2.0	2.0	0.0	0.0
0.1	0.998657928234	0. 998657928226	0.9124×10^{-11}	1.7×10^{-05}
0.2	0.603793035989	0. 603793035967	2.2106×10^{-11}	1.6×10^{-05}
0.3	0.481435906578	0. 481435906574	0.3921×10^{-11}	9.3×10^{-06}
0.4	0.481524407956	0. 481524407924	3.1242×10^{-11}	4.6×10^{-06}
0.5	0.536631277777	0. 536631277724	5.2046×10^{-11}	1.8×10^{-06}
0.6	0.616459494098	0. 616459494074	2.3262×10^{-11}	4.2×10^{-07}
0.7	0.707395727432	0. 707395727421	1.1310×10^{-11}	1.8×10^{-06}
0.8	0.803323114546	0. 803323114523	2.2616×10^{-11}	2.3×10^{-06}
0.9	0.901493171616	0. 901493171601	1.5367×10^{-11}	3.8×10^{-07}

5.0 Conclusion.

From the results of the numerical implementation of the method when adopted to solve initial value problems of first order ordinary differential equations, the method proposed in this paper are consistent, convergent and can compete favourably with existing methods.

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