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A New One - Step Hybrid Stomer Cowell Type Method for the Numerical Solution of the Initial Value Problems of General Second Order Ordinary Differential Equations

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Abstract

In this paper, a new one-step hybrid method for direct solution of general second order ordinary differential equations is developed. Power series is adopted as the basis function. The differential systems arising from the basis function are collocated at the entire selected grid and off grid points while the approximate solutions are interpolated at two points nearest to the end points to generate a One – step hybrid Stomer cowell type numerical method. To provide the starting values for the numerical implementation of the method, the one-step scheme is expanded term by term, by Taylor's series expansion while f', f'' and f''' are calculated by the use of partial derivative techniques. The resulting numerical method is analyzed for its basic properties and it was discovered that it was zero stable, consistent, and convergence. The efficiency of the method is tested on some test problems and the accuracy compared with the existing methods. The results give better accuracy over the existing method.

Keywords: One-Step Method, Second Order Initial Value Problems, Ordinary Differential Equations, Taylor's Series, Hybrid Methods.

1.0 Introduction

Varieties of problems arising from the fields of science, engineering, social science and so on are modelled using differential equations. These problems range from first order to higher order problems like falling body problems, orthogonal trajectories, damped mechanical oscillator, electrical circuit problems, and so on. In this work, we considered a general second order initial value problem (IVP) of the form:

$$y'' = f(x, y, y'), y(a) = y_0, y'(a) = \eta \quad (1.1)$$

Conventionally, problems of the form (1.1) are solved by first reducing it to systems of first order ordinary differential equations of the form:

$$y' = f(x, y), y(x_0) = y_0, \qquad x, y \in R^n, f \in C'[a, b] \quad (1.2)$$

The resulting equations are then solved by any appropriate numerical or analytical methods. This is widely discussed in the Literature [1- 4]. This approach is considered uneconomical as a result of computational burdens, human and computer time wastage [5]. A special class of second order differential equation:

$$y'' = f(x, y) \quad (1.3)$$

has been considered by several scholars [3, 6, 7]. Lambert [3] proposed the shooting method which essentially converts Boundary Value Problems into Initial Value Problems of the form (1.3).

However, many eminent scholars have developed methods of solutions of (1.1) without reduction to systems of first order ODES [2, 8 – 11]. They proposed Linear Multistep Methods (LMM) with continuous coefficients for IVPs of the form (1.1) in the predictor-corrector mode based on collocation method using power series polynomial as the basis function and Taylor's series algorithm to supply the starting values.

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Hybrid method is another method which has been developed by many scholars. For instance [12] developed a P-stable, one-leg hybrid LMM in which Pade approximation was used as a basis function for solving (1.1). Other scholars [13 – 16] have also developed hybrid methods.

Adesanya [17] adopted a method of collocation and interpolation to develop a continuous LMM which is evaluated at different grid points to give discrete methods and then adopted block methods approach in the discrete methods to generate independent solutions. Anake [18] proposed hybrid one-step block methods that can solve (1.1) directly. Bolaji [19] developed and implemented implicit hybrid block methods for the numerical integration of third order ordinary differential equations.

Though some of the above mentioned authors have made use of Taylor's series to supply starting values since the methods are not self starting, little has been said about one-step hybrid methods with the use of Taylor's series as the major method of implementation. Consequently, this work is aimed at developing one-step hybrid methods in the Taylor's series mode to solve directly equation (1.1).

2.0 Derivation of The Method

An approximate solution to the Initial Value Problems of the kind (1.1) is proposed in the form:

$$y(x) = \sum_{j=0}^{(m+n)-1} a_j x^j \tag{2.1}$$

which is a power series with a single variable x and $(m+n)$ is the sum of the collocation and interpolation points.

We take the first and second derivatives of (2.1) as follows:

$$y'(x) = \sum_{j=1}^{(m+n)-1} j a_j x^{j-1} \tag{2.2}$$

$$y''(x) = \sum_{j=2}^{(m+n)-1} j(j-1) a_j x^{j-2} \tag{2.3}$$

By combining (1.1) and (2.3) we have the differential system:

$$\sum_{j=2}^{(m+n)-1} j(j-1) a_j x^{j-2} = f(x, y, y') \tag{2.4}$$

By collocating (2.4) at selected grid and off grid points, $x = x_{n+i}, 0 \le i \le 1$ and interpolating (2.1) at selected grid and off grid points result into a system of equations:

$$\sum_{j=2}^{(m+n)-1} j(j-1) a_j x^{j-2} = f_{n+i} \tag{2.5}$$

and $$\sum_{j=0}^{(m+n)-1} a_j x^j = y_{n+i} , \ 0 \le i \le 1 \tag{2.6}$$

Where $x_{n+i} = x_n + ih$ (2.7)

Which when solved for a_j's yield a method expressed in the form:

$$y(x) = \sum_{j=0}^{k} \alpha_j(x) y_{n+j} + \sum_{j=0}^{k} \beta_j(x) f_{n+j} \tag{2.8}$$

Where $k = 1$ and $f_{n+j} = f\left(x_{n+j}, y_{n+j}, y^1_{n+j}\right)$, $0 \le j \le 1$

The coefficients $\alpha_j(x)$ and $\beta_j(x)$ in Equation (2.8) are as follows if

$$t = \frac{x - x_n}{h} \tag{2.9}$$

$$\alpha_0(t) = -2t + 1$$

$$\alpha_{1/2}(t) = 2t$$

$$\beta_0(t) = \frac{h^2}{48}\left(-7t + 24t^2 - 24t^3 + 8t^4\right)$$

$$\beta_{\frac{1}{2}}(t) = \frac{h^2}{24}\left(-3t + 16t^3 - 8t^4\right)$$

$$\beta_1(t) = \frac{h^2}{48}\left(t - 8t^3 + 8t^4\right) \qquad (2.10)$$

Taking the first derivative of (2.10) and evaluating the results together with (2.10) at t=1 which implies that $x = x_{n+1}$ gives our scheme:

$$y_{n+1} = 2y_{n+\frac{1}{2}} - yn + \frac{h^2}{48}\left(f_{n+1} + 10 f_{n+\frac{1}{2}} + f_n \right) \qquad (2.11)$$

The scheme is of order $C_4 = 4$ and error constant $C_6 = \frac{-1}{15360}$ and interval of absolute stability $X(\theta) = (-9.6, 0)$.

The first derivative is given by:

$$y'_{n+1} = \frac{2}{h}\left(y_{n+1/2} - y_n \right) + \frac{h}{48}\left(9 f_{n+1} + 26 f_{n+1/2} + f_n \right) \qquad (2.12)$$

3.0 Implementation of the Method.

To implement the method, there is a need for the provision of starting values for the scheme (2.11), thus Taylor's series algorithm is used to generate y values for the approximate solution, by expanding term by term the scheme and its first derivative up to the order of the scheme, as follows:

$$y_{n+i} = y(x_n + ih) = y_n + ihy'(x_n) + \frac{(ih)^2}{2!} f_n + \ldots \qquad (3.1)$$

$$y'_{n+i} = y'(x_n) + ihf_n + \frac{(ih)^2}{2!} f'_n + \ldots \qquad (3.2)$$

and

$$f_{n+i} = y''(x_n + ih) = f_n + ihf'_n + \frac{(ih)^2}{2!} f''_n + \ldots \qquad (3.3)$$

Where

$$f_n = f\left(x_n, y_n, y'_n\right)$$

$$f^{(i)} = f^{(i)}\left(x_n, y_n, y'_n\right), i = 1,2,3$$

We find f', f'' and f''' by the use of partial derivative techniques as follows:

$$f' = \frac{df}{dx} = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + f\,\frac{\partial y}{\partial y'}$$

Where (3.4)

$$f'' = \frac{d^2 f}{dx^2} = 2\left(Ay' + Bf\right) + Cf_{y^1} + D + E$$

$$A = \frac{\partial^2 f}{\partial x \partial y} + f\,\frac{\partial^2 f}{\partial y \partial y'} \qquad B = \frac{\partial^2 f}{\partial x \partial y'}$$

$$C = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + f\,\frac{\partial f}{\partial y'} \qquad D = \frac{\partial^2 f}{\partial x^2} + (y')^2\frac{\partial^2 f}{\partial y^2} + f^2\,\frac{\partial^2 f}{(\partial y')^2}$$

$$E = f\,\frac{\partial f}{\partial y}$$

(3.5)

and

$$f''' = \frac{d^3 f}{dx^3} = 2G + 3(Hy' + If) + Jf_{y^1} + K + L + M$$

Where

$$G = y^1 f^1 \frac{\partial^2 f}{\partial y \partial y^1} + f^1 \frac{\partial^2 f}{(\partial y^1)^2} + y^1 ff_{y^1} \frac{\partial^2 f}{\partial y \partial y^1} + f^1 \frac{\partial^2 f}{\partial x \partial y^1}$$

$$H = \frac{\partial^3 f}{\partial x^2 \partial y} + y^1 \frac{\partial^3 f}{\partial x \partial y^2} + f \frac{\partial^2 f}{\partial y^2} + y^1 f \frac{\partial^3 f}{\partial y^2 \partial y^1} + f^2 \frac{\partial^3 f}{\partial y (\partial y^1)^2} + 2 \frac{\partial^3 f}{\partial x \partial y \partial y^1}$$

$$I = \frac{\partial^3 f}{\partial x^2 \partial y^1} + \frac{\partial^2 f}{\partial x \partial y} + f \frac{\partial^3 f}{\partial x (\partial y^1)^2} + f \frac{\partial^2 f}{\partial y \partial y^1} \tag{3.6}$$

$$J = f \frac{\partial f}{\partial y} + 2y^1 \frac{\partial^2 f}{\partial x \partial y} + (y^1)^2 \frac{\partial^2 f}{\partial y^2} + f^1 \frac{\partial f}{\partial y^1} + 2f \frac{\partial^2 f}{\partial x \partial y^1} + f^2 \frac{\partial^2 f}{(\partial y^1)^2}$$

$$K = \left(\frac{\partial f}{\partial x} + y^1 \frac{\partial f}{\partial y} + f \frac{\partial f}{\partial y^1} \right) \left[\frac{\partial^2 f}{\partial x \partial y^1} + y^1 \frac{\partial^2 f}{\partial y \partial y^1} + f \frac{\partial^2 f}{(\partial y^1)^2} \right]$$

$$L = \frac{\partial^3 f}{\partial x^3} + f^3 \frac{\partial^3 f}{(\partial y^1)^3} + (y^1)^3 \frac{\partial^3 f}{\partial y^3} \quad and \quad M = f^1 \frac{\partial f}{\partial y}$$

The scheme (2.11) alongside the Taylor's series algorithm for the supply of its starting values was translated to computer codes using FOTRAN programming language for its computer implementation on a digital computer using test problems towards determining the accuracy of the method.

Test problem 1

We consider the initial value problem:

$$y'' - x(y')^2 = 0 \quad y(0) = 1, y'(0) = \frac{1}{2} \qquad h = \frac{0.1}{32} \tag{3.7}$$

Whose exact solution is:

$$y = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right) \tag{3.8}$$

The numerical solution to the problem was compared with [8] of order 6 with step length k = 4, which was implemented in a predictor-corrector mode. The results are as shown in Table i.

Test Problem 2

We consider the initial value problem:

$$y'' = y' \quad y(0) = 0, y'(0) = -1 \quad \text{with } h = 0.1 \tag{3.9}$$

Whose exact solution:

$$y(x) = 1 - \exp(x) \tag{3.10}$$

The numerical solution to the problem was compared with the method of [18] of order 7 with step length k = 1 which was implemented in block mode. The results are as shown in Table ii.

4.0 Conclusion

In this paper, we have discussed the derivation and implementation of a new One – step hybrid numerical method for the direct solution of initial value problems of general second order ordinary differential equations.

The implementation strategy which is based on the use of Taylor's series algorithm has enabled derivatives of continuous schemes - to any possible order - to be computed. This allows direct solution of Initial Value Problems of ordinary differential equations without reduction to systems of first order differential equations.

The accuracy of the derived method is tested with two test problems and the results were compared with [8] of order 6 with step length k = 4, which was implemented in a predictor-corrector mode and [18] of order 7 with step length k = 1 which was implemented with block method. Our new method gives better accuracy.

Table i: Results and errors for problem (3.7)

x	Exact Solution	New Results	Error	Error in [8]
0.2	1.100335347731	1.100335347732	4.2477e – 013	0.1982e - 008
0.4	1.202732554054	1.202732554056	1.7908e – 012	1.7908e - 012
0.6	1.309519604203	1.309519604208	4.7622e – 012	4.7622e - 012
0.8	1.423648930194	1.423648930204	0.1569e – 006	1.0762e - 011
1.0	1.549306144334	1.549306144358	0.3880e – 006	2.4050e - 011

Table ii: Results and errors for problem (3.9)

x	Exact Solution	New Results	Error	Error in [18]
0.2	-0.221402758160	-0.221402746312	1.1848e-008	0.8464e - 006
0.4	-0.491824697641	-0.491824672331	2.5310e-008	0.4504e - 005
0.6	-0.822118800391	-0.822118759872	4.0519e-008	0.1199e - 004
0.8	-1.225540928493	-1.225540870904	5.7588e-008	5.7588e - 008
1.0	-1.718281828459	-1.718281751861	7.6598e-008	7.6598e - 008

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