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Application of homotopy perturbation method to find an analytical solution on some linear and non-linear differential Equations

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Abstract

In this article, the problem of linear and non-linear differential equations is presented. The homotopy perturbation method (HPM) is employed to solve both linear and non-linear differential equations, with help of embedding parameters which is used to control the convergence. This new technique was used to obtain solutions that are very close to the exact solution of the equation. The obtained results have been compared with the exact solution of the differential equation and with numerical solution for those equations that does not have exact solution. The results reveal that the method is tremendously effective.

Keywords: Homotopy Perturbation method, differential Equations, Analytical Solution, Numerical Solution.

1.0 Introduction

It is well known and documented fact that many phenomena in engineering, science, medicine control etc. can be modeled using the theory of derivatives and integrals. Also it is noted that solution to most differential equations that arise from these models cannot be easily obtained by analytical means. Common analytical procedures linearize the system or assume the non-linearities are relatively insignificant. Such procedures change the actual problem to make it tractable by the convection methods. These changes, sometimes, affect seriously the solution. Hence, much attention has been devoted to the search for reliable and more efficient solution methods for equation modeling physically in science; engineering etc. one of the methods is the Homotopy perturbation method. The homotopy perturbation method (HPM) was introduced by He [1-4] in 1998. In this method the solution is considered as the summation of an infinite series which usually converges rapidly to the exact solution. The essential idea in this method is to introduce a homotopy parameter, say P which takes the values from 0 to 1. For P =0, the system of equations takes a simplified form. Which readily admits a particular simple solution, when p is gradually increased to 1, the system goes through a sequence of deformations, the solution of each of which is close to that at the previous stage of deformation. Eventually at P=1, the system takes the original form of equation and the final stage of deformation gives the desire solution. We must remark here that Liao [5, 6] keeps sufficient room for experimenting with the convergence of the approximations by introducing auxiliary parameters and also at times, one auxiliary non-zero function. The radius of convergence of the solution can be often dramatically enlarged by choosing appropriate values of the above mentioned parameter and function.

In this study, a powerful analytical method known as HPM is used to solve linear and non-linear differential equation arising from modeling physical phenomena. In addition, the results of this method are compared with exact, numerical solution. It will be shown that the results obtained with HPM just with a few terms in series expansion, are in good agreement with both numerical and exact solution.

2.0 Homotopy Analysis Method

In what follows, a description of the homotopy analysis method as it appears in various literatures He[1, 2] will be presented. Consider a non-linear differential equation

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{1}$$

The general boundary condition is

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma \tag{2}$$

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Where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω. Based on the homotopy perturbation method, the operator A can be generally divided into two parts of L and N, where L is the linear part, while N is the non-linear one. Equation (1) can therefore be written as

$$L(u)+N(u)-f(r)=0 \qquad (3)$$

By the homotopy technique, we construct a homotopy as

$V(r,p):\Omega X[0,1]\rightarrow R$ which satisfies

$$H(v,p)=(1-p)[L(v)-L(u_0)]+P[A(v)-f(r)]=0 \quad P\in[0,1], r\in\Omega \qquad (4)$$

Where $P\in[0,1]$ is an embedding parameter and u_0 is an initial or first approximation of equation (2) which satisfies the boundary conditions. The process of the changes in P from zero to one is that of $V(r,p)$) changing from and u_0 to u. In topology it is called deformation.

Obviously considering equation (4) it is clear that

$$H(r,0)=L(v)-L(u_0)=0$$
$$H(r,1)=A(v)-f(r)=0 \qquad (5)$$

Equation (5) is referred to as homotopy

By considering V as

$$V(r)=V_0(r)+pV_1(r)+p^2V_2(r)+\cdots \qquad (6)$$

the best approximation is

$$u=\lim_{p\rightarrow 1}V=V_0(r)+V_1(r)+V_2(r)+\cdots \qquad (7)$$

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which eliminated limitations of the traditional perturbation techniques. The series (7) is convergent for more cases. Some criteria are suggested for convergence of the series in He [1].

3.0 Application of homotopy

In this section, we demonstrate the main algorithm of homotopy perturbation method on linear and nonlinear differential equation with initial condition, namely we consider.

Example 1

$$y^1 = y \qquad y(0)=1 \qquad (8)$$

According to H P M, the homotopy construction of equation (8) can be expressed as follows:

$$(1-p)[\frac{dv}{dx}-\frac{dy_0}{dx}]+p[\frac{dv}{dx}-v]=0 \qquad (9)$$

The boundary condition is

$$v(0)=y(0)=1 \qquad (10)$$

Substituting equation (6) into equation (9) and equating the coefficients of like powers P, we get the following set of differential equations

$$p^0: \frac{dv_0}{dx}-\frac{dy_0}{dx}=0$$
$$p^1: \frac{dv_1}{dx}-v_0=0$$
$$p^2: \frac{dv_2}{dx}-v_1=0$$
$$p^3: \frac{dv_3}{dx}-v_2=0 \qquad (11)$$
$$p^4: \frac{dv_4}{dx}-v_3=0$$
$$p^5: \frac{dv_5}{dx}-v_4=0$$

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Solving equation (11), we obtain

$$v_0 = 1$$
$$v_1 = x$$
$$v_2 = \frac{x^2}{2!}$$
$$v_3 = \frac{x^3}{3!}$$
$$v_4 = \frac{x^4}{4!}$$
$$v_5 = \frac{x^5}{5!}$$

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Therefore from the results we can obtain

$$v(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... + \frac{x^n}{n!} \tag{12}$$

$$\lim_{n\to\infty} \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

which is the exact solution of the problem. The absolute error for various values of x and M(number of terms) are tabulated in Table 1

Table 1: Comparism of exact solution with Homotopy Perturbation method (HPM) for various values of X and M (number of terms) for test problem.

	M=10		
X	H.P.M	EXACT	ERROR
0	1.000000000	1.000000000	0
0.2	1.221402758	1.221402758	0
0.4	1.491824698	1.491824698	0
0.6	1.822118800	1.822118800	0
0.8	2.225540926	2.225540928	0.000000002
1	2.718281801	2.718281828	0.000000027
	M=20		
X	H.P.M	EXACT	ERROR
0	1.000000000	1.000000000	0
0.2	1.221402758	1.221402758	0
0.4	1.491824698	1.491824698	0
0.6	1.822118800	1.822118800	0
0.8	2.225540928	2.225540928	0
1	2.718281828	2.718281828	0

Example 2:

$$y^{111} = 3\sin x \qquad y(0) = 1 \qquad y^{1}(0) = 0 \qquad y^{11}(0) = -2 \tag{13}$$

The exact solution is

$$y(x) = 3\cos x + \frac{1}{2}x^2 - 2$$

According to HPM, the homotopy construction of equation (13) can be expressed as follows:

$$(1-p)[\frac{d^3v}{dx^3}-\frac{d^3y_0}{dx^3}]+\ p[\frac{d^3v}{dx^3}-3\sin x]=0 \quad (14)$$

The boundary condition is

$$\begin{aligned} &v(0) = y(0) = 1 \\ &v^1(0) = y^1(0) = 0 \\ &v^{11}(0) = y^{11}(0) = -2 \end{aligned} \quad (15)$$

Substituting equation (6) into equation (14) and equating the coefficients of like powers P, we get the following set of differential equations.

$$\begin{aligned} &p^0: \frac{d^3v_0}{dx^3} - \frac{d^3y_0}{dx^3} = 0 \\ &p^1: \quad \frac{d^3v_1}{dx^3} \quad + \frac{d^3y_0}{dx^3} \quad -3\sin x = 0 \\ &p^2: \quad \frac{d^3v_2}{dy^3} = 0 \end{aligned} \quad (16)$$

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First for simplicity we take

$$v_0 = y_0$$

In present work we start the iteration by defining y_0 as a Taylor series of order two near x=0, so that it could be resulted in highly accurate solution near x= 0.i.e

$$v_0 = y_0 = y(0) + y^1(0)x + \frac{1}{2}y^{11}(0)x^2 \quad (17)$$

By applying boundary conditions equation (15) and solving equation for v_1, v_2 we derive

$$\begin{aligned} &v_0 = -x^2 + 1 \\ &v_1 = 3\cos x + \frac{3}{2}x^2 - 3 \\ &v_2 = 0 \end{aligned} \quad (18)$$

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According to equation (6) and the assumption P=1, we get

$$v = 3\cos x + \frac{3}{2}x^2 - 3 + 1 - x^2 + 0$$

$$y = \lim_{p\to 1} v = 3\cos x + \frac{1}{2}x^2 - 2 \quad (19)$$

Equation (19) is the exact solution of the problem.

Example 3

$$y^{11} = (y^1)^2 \qquad y(0) = 1 \qquad y^1(0) = \frac{1}{2} \quad (20)$$

According to HPM, the homotopy construction of equation (20) can be expressed as follows:

$$(1-p)[\frac{d^2v}{dx^2}-\frac{d^2y_0}{dx^2}]+\ p[\frac{d^2v}{dx^2}-x(v^{'})^2]=0 \quad (21)$$

The boundary condition

$$v(0) = y(0) = 1$$

$$v^1(0) = y^1(0) = \frac{1}{2} \qquad (22)$$

Substituting equation (6) into equation (21) and equating the coefficients of like powers P, we get the following differential equations.

$$p^0: \frac{d^2v_0}{dx^2} - \frac{d^2y_0}{dx^2} = 0$$

$$p^1: \frac{d^2v_1}{dx^2} + \frac{d^2y_0}{dx^2} - x(v_0^1)^2 = 0$$

$$p^2: \frac{d^2v_2}{dx^2} - 2xv_0^1v_1^1 = 0 \qquad (23)$$

$$p^3: \frac{d^2v_3}{dx^2} - 2xv_0^1v_1^1 - x(v_1^1)^2 = 0$$

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We start the iteration by defining y_0 as a Taylor series of order one near $x = 0$. So that

$$v_0 = y_0 = y(0) + y^1(0)x \qquad (24)$$

By applying the boundary condition (22) and solving equation (23) for v
We derive

$$v_0 = \frac{1}{2}x + 1$$

$$v_1 = \frac{x^3}{2^3.3} \qquad (25)$$

$$v_2 = \frac{x^5}{2^5.5}$$

$$v_3 = \frac{x^7}{2^7.7}$$

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According to equation (6) and assuming P=1 we have

$$v(x) = 1 + \frac{1}{2}x + \frac{x^3}{2^3.3} + \frac{x^5}{2^5.5} + \frac{x^7}{2^7.7} + ... + \frac{x^{2n+1}}{2^{2n+1}.(2n+1)}$$

$$v(x) = 1 + \frac{1}{2}x + \sum_{n=1}^{\infty} \frac{x^{2n+1}}{2^{2n+1}.(2n+1)} \qquad (26)$$

Table 2: comparison of the homotopy method (HPM) with numerical solution(NS) for various values of X and M (number of terms) for test problem

	M=10		
X	H.P.M	NS(RKF45)	ERROR
0	1.000000000	1.000000000	0
0.2	1.100335348	1.100335100	0.0000002471
0.4	1.202732554	1.202731845	0.0000007080
0.6	1.309519604	1.309519501	0.0000001030
0.8	1.423648930	1.423649485	-0.0000005551
1	1.549306137	1.549307551	-0.0000014143
	M=20		
X	H.P.M	NS	ERROR
0	1.000000000	1.000000000	0
0.2	1.100335348	1.100335100	0.00000024709
0.4	1.202732554	1.202731845	0.0000007080
0.6	1.309519604	1.309519501	0.0000001030
0.8	1.423648930	1.423649485	-0.0000005551
1	1.549306144	1.549307551	-0.0000014076

4. 0 Conclusion

In this article, HPM was used to obtain an analytical solution for linear and non-linear differential equations. It was shown by choosing an appropriate linear operator that just a few orders in series expansion are sufficient to obtain an accurate solution that is valid for the whole domain of the solution. Also , this method is conceptually very simple and has almost globally convergent characteristics. Based on this advantage, in this method we don't need to control the value of the error as in the classical methods like Bisection, Newton-Raphson method etc. in particular, the proposed method is especially attractive compared with existing methods. Finally, some remarkable virtues of the introduced method were illustrated through some examples.

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