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Analysis of FTFS Computational Scheme for the Solution of First Order Partial Differential Equations

Odio Augustine Onyejiuwa

Department of Mathematics
University of Nigeria, Nsukka

Abstract

We consider a one dimensional first order partial differential equation.

The Equation arises as a simplification of the general transport equation of fluid flow in boundary layer problem. The equation is obtained by setting the source and diffusion term to zero. We apply the FTFS or forward in time and forward in space scheme for the first order partial differential equation. The scheme is expanded by the Taylor series method and truncated to the first two terms. The discretized equation satisfied the numerical properties such as consistency, stability and convergence. The bane of this work is based on the theorem of Lax. Numerical solution gotten here are expected to converge weakly due to truncation error. Here there exists light perturbation of solution unlike solution of second order partial differential equation.

Keywords: Transport equation, Boundary layers problem, Computational Scheme, Consistency, Stability, Convergence, Lax Theorem.

1.0 Introduction

We consider the forward in time and forward in space (FTFS) scheme given as

$$u_i^{n+1} = u_i^n - \beta(u_i^n - u_{i+1}^n) \qquad (1)$$

Where β is any constant for the one dimensional wave equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}$$

where, at

$$t = 0 \qquad u(x,0) - f(x),\ 0 \le x \le 1$$

as

$$x = 0,\ u(0,t) = g(x),\ t \ge 0 \qquad (2)$$

$\frac{\partial u}{\partial x}$ is a differential operator in u, which contains only partial derivatives with respect to the space coordinate $x_1, x_2, x_3, \ldots, x_p$ and time variables. Equation (2) is a simplification of the general transport equation and is obtained by putting the diffusion and the source term to zero in the boundary layers problem [1]

In general, the unknown $u(x,t) = u(x_1, x_2 \ldots x_p, t)$ may be either a scalar or vector function. The solution of (2) is required in an arbitrary region $\Re x\ [0, T]$ with suitable boundary condition $\partial\Re x[0,T]$ where $\Re$ is normally a closed region in $x_1, x_2, \ldots x_p$ space, $\partial\Re$ is the boundary of $\Re$, and [0, T] is the time interval [2]. The u_i^n in equation [1] refers to $u(x,t) = u(i\Delta x, n\Delta t)$, that is value of the variable u at the i^{th} space location, n^{th} is the time step and $\beta = \frac{c\Delta t}{\Delta x}, where \Delta x$ and Δt are the increment in space and time, c is the speed of wave [3]. Scheme (1) is explicitly in time and allow marching forward in time from given initial conditions [4].

Corresponding author: E-mail: augustine.odio@yahoo.com., Tel.: +2348062976038

A formal analysis procedure has been developed to deal with the important question on how accurate a numerical solution is compared to the exact solution of the differential equation. These analysis has been enacted in the equivalence theorem of Lax.

(Equivalence theorem of Lax [6]): For a well posed linear, initial, value problem with a consistent discretization stability is the necessary and sufficient condition for convergence of the numerical scheme. The three properties mention must be studied to satisfy Lax theorem.

2.0 Main Result

2.1 Consistency

Consistence requires that the discretized, equations should tend to the original differential equations when Δt and Δx tend to zero. Consistency is ensured if the truncation errors goes to zero as Δt and Δx tend to zero. We demonstrate this with an example. Consider the forward in time and forward in space scheme

$$u_i^{n+1} = u_i^n - \beta(u_i^n - u_{i+1}^n) \tag{3}$$

Using Taylor series expansion [5] u_i^{n+1} and u_{i+1}^n can be expanded as follows

as

$$u_i^{n+1} = u_i^n + \Delta t(u_t)_i^n + \frac{\Delta t^2}{2}(u_{tt})_i^n + \tag{4}$$

$$u_{i+l}^n = u_i^n + \Delta x(u_x)_l^n + \frac{\Delta x^2}{2}(u_{xx})_l^n + \frac{\Delta t^3}{6}(u_{xxx})_i^n + ... \tag{5}$$

Substituting these in equation (3) and simplifying, we get

$$u_i^n + \Delta t u_i^n + \frac{(\Delta t)^2}{2}u_{tt}^n = u_i^n - \beta(u_i^n - u_i^n - \Delta x u_x^n - \frac{(\Delta x)^2}{2}u_{xx}^n \ldots)$$

$$= u_i^n - \beta u_i^n + \beta u_i^n + \beta\Delta x u_x^n + \frac{(\Delta x)^2}{2}\beta u_{xx}^n$$

$$= u_i^n + \Delta x\beta u_x^n + \frac{(\Delta x)^2}{2}\beta u_{xx}^n - ...$$

$$u_i^n + \Delta t u_i^n + \frac{(\Delta t)^2}{2}u_{tt}^n = u_i^n + \Delta x\beta u_x^n + \frac{(\Delta x)^2}{2}\beta u_{tt}$$

$$\Delta t u_t^n + \frac{(\Delta t)^2}{2}u_{tt}^n = \Delta x\beta u_x^n + \frac{(\Delta x)^2}{2}\beta u_{xx}$$

$$\Delta t u_t^n = \Delta x\beta u_x^n + O((\Delta t),^2 (\Delta x)^2)$$

where β $\frac{\Delta x}{\Delta t}$ is a constant which is equivalent to c

It implies that the FTFS or forward in time and forward in space scheme is consistent. The truncation error on the right hand side is of the first order in both time and space and that it varnishes as the limit of Δt and Δx tending to zero. Thus the forward in time and forward in space scheme satisfies the consistency condition.

2.2 Stability

A numerical solution method is said to be stable if it does not amplify the errors (due to rounding off, truncation errors, mistakes etc.) that appear in the course of a numerical solution. For time-dependent problems, stability guarantees that the method produces a bounded solution if the exact solution itself is bounded. Thus, a condition for stability can be formulated by the requirement that any error ε_i^n between the computed solution u and the exact solution of the difference equation $\bar{u}$ should remain uniformly bounded as $n \to \infty$ at fixed Δt. If we define the error ε as the difference between the computed solution and the exact solution of the discretized equation

$$\varepsilon_i^n = u_i^n - \bar{u}_i^n \tag{6}$$

the stability condition can be written as

$$\lim_{n \to \infty} \left| \varepsilon_i^n \right| \le K \text{ at fixed } \Delta t \tag{7}$$

with K being independent of n. Equation (7) does not guarantee that the error does not become unacceptably large for intermediate values of $t_n = n\,\Delta t$ and a more general definition of stability requires that any component of the initial solution is not amplified without bounds.

2.3 Convergence

A numerical scheme is said to be convergent if the computed solution of the discretized equations tends to the exact solution of the differential equation as the grid and time spacing tend to zero. Formally, this can be defined as follows. The computed solution u_i^n should approach the exact solution $\bar{u}(x,t)$ of the differential equation at any point $x_i = i\,\Delta x$ and $t_n = n\,\Delta t$ when Δx and Δt tend to zero while allowing x_i and t^n constant. It follows that, the error

$$\varepsilon_i^n = u_i^n - \bar{u}(i\Delta x, n\Delta t) \tag{8}$$

satisfies the following convergence condition:

$$\lim_{\Delta t, \Delta x \to 0} \left| \varepsilon_i^n \right| \to 0 \text{ at fixed } x_i = i\Delta x \text{ and } t_n = n\Delta t. \tag{9}$$

This tell us that, a solution which does not change significantly with further decrease in the grid and time step spacing, is the correct solution of the differential equation.

Conclusion

In this work, we make the diffusion and source term of the transport equation in the boundary layers problem to be zero in order to obtain the one dimensional first order partial differential equation. Furthermore, we also show that, the numerical scheme satisfied the consistency, stability and convergence properties of the work.

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