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Application of Orthogonality of the Trigonometric System to Fourier Series

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Abstract

The key to the Euler formulas is the orthogonality of the trigonometric system, which is a concept of basic importance. In this paper, we shall be looking at the derivation of these Euler formulas using the orthogonality of the trigonometric system. This concept avoided all those lengthy integral techniques that are usually encountered with series of functions; to produce a short and concise result.

1.0 Introduction

A function $f(x)$ is called a periodic function if $f(x)$ is defined for all real x and if there is some positive number p, called a period of $f(x)$ such that;

$$f(x+p) = f(x), \quad \text{for all } x \tag{1}$$

Previous works has been carried out on how the various functions $f(x)$ of period 2π can be represented in terms of the simple functions.

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots \tag{2}$$

All these functions have the period 2π and they constitute the so called trigonometric system.
The series to be obtained will be a trigonometric series given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{3}$$

Equation (3) is called the Fourier series of $f(x)$where the coefficient are called the Fourier coefficients of $f(x)$ given by the Euler formula

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{dx} \tag{4a}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \qquad n = 1, 2, \ldots \tag{4b}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \qquad n = 1, 2, \ldots \tag{4c}$$

Basic examples

Before we derive the Euler formulas (4), let us become familiar with the application of (3) and (4) in the case of an important example.
Periodic rectangular wave
(i) Find the Fourier coefficient of the periodic function $f(x)$
Where;

$$f(x) = \begin{cases} -k & if -\pi < x < 0 \\ k & if\ 0 < x < \pi \end{cases}$$

And $f(x+2\pi) = f(x)$

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Solution

From (4a) we obtain $a_0 = 0$. this can also be seen without integration, since the area under the curve of $f(x)$ between $-\pi$ and π is zero.

From (4b)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-k\,\frac{\sin n\pi}{n}\,\Big|_{-\pi}^{0} + k\,\frac{\sin nx}{n}\,\Big|_{0}^{\pi}\right] = 0$$

Because $\sin nx = 0\ at - \pi, 0\ and\ \pi\ for\ all\ n = 1, 2, \ldots$

Similarly, from (4c), we obtain

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\,, = \frac{1}{n}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[k\,\frac{\cos nx}{n}\,\Big|_{-\pi}^{0} + k\,\frac{\cos nx}{n}\,\Big|_{0}^{\pi}\right]$$

Since $\cos(-\propto) = \cos\propto\ and \cos 0 = 1, this\ yields$

$$b_n = \frac{k}{n\pi}\left[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0\right] = \frac{2k}{n\pi}(1 - \cos n\pi)$$

Now, $\cos\pi = -1, \cos 2\pi = 1, \cos 3\pi = -1, etc;$

$$\cos n\pi = \begin{cases} -1 & for\ \ odd\ \ n \\ 1 & for\ even\ n \end{cases}$$

And thus, $1 - \cos n\pi = \begin{cases} 2 & for\ odd\ \ n \\ 0 & for\ even\ n \end{cases}$

Hence, the Fourier coefficients b_n of our function are;

$$b_1 = \frac{4k}{\pi}, \qquad b_2 = 0, \qquad b_3 = \frac{4k}{3\pi}, \qquad b_4 = 0, \qquad b_5 = \frac{4k}{5\pi}$$

Since the a_n are zero, the Fourier's series of $f(x)$ is

$$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

The partial sums are

$$S_1 = \frac{4k}{\pi}\sin x,\ S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right), etc$$

Furthermore, assuming that $f(x)$ is the sum of the series and setting $x = \frac{\pi}{2}$, we have;

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} + \ldots\right)$$

Thus

$$1 - {}^{1}/_{3} + {}^{1}/_{5} - {}^{1}/_{7} + \ldots = \frac{\pi}{4}$$

(ii) Find the Fourier series expansion of the periodic function of period 2π.

$$f(x) = x^2, \quad -\pi \le x \le \pi$$

Hence, find the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$

Solution

$$f(x) = x^2, \quad -\pi \le x \le \pi$$

This is an even function. $\therefore\ b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi^2\sin nx}{n}\right) + \left(\frac{2\pi\cos nx}{n^2}\right) - \left(\frac{2\sin n\pi}{n^3}\right)\right] = \frac{(-1)^n}{n^2}$$

Fourier series is

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots a_n \cos nx + \cdots$$

$$x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\right]$$

On setting $x = 0$, we have

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots\right]$$

$$or \qquad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots = \frac{\pi^2}{12}$$

This is a famous result obtained by leibniz [1] in 1673 from geometric consideration. It illustrates that the values of various series with constant terms can be obtained by evaluating Fourier series at specific points

2.0 Orthogonality Of The Trignometric System

Theorem 1

The trigometric system (2) is orthogonal on the interval $-\pi \le x \le \pi$ (hence also on $0 \le x \le 2\pi$ or any other interval of length 2π because of periodicity): that is, the integral of the product of any two functions of (2) over that interval is 0, so that for any integers n and m.

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \qquad (n \ne m) \qquad (5a)$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \qquad (n \ne m) \qquad (5b)$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \qquad (n \ne m) \quad or\ n = m \qquad (5c)$$

Proof

This follows simply by transforming the integrands trigonometrically from product into sum [2], [3] in (5a) and (5b).

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x \, dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x \, dx$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x \, dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x \, dx$$

Since $m \ne n$, the integrals in the right are all 0

Similarly, in (5c) for all integer

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)x \, dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin(n-m)x \, dx$$

$$= 0 + 0 = 0$$

3.0 Derivation of the Euler Formulas

3.1 Application of Theorem 1 to the Fourier Series (3)

We prove (4a)

Integrating on both sides of (3) from $-\pi\ to\ \pi$, we get

$$\int_{-\pi}^{\pi} f(x)\, dx = \int_{-\pi}^{\pi}\left[a_o + \sum_{n-1}^{\infty}(a_n \cos nx + b_n \sin nx\right] dx$$

We now assume that term wise integration is allowed

$$= a_o \int_{-\pi}^{\pi} dx + \sum_{n-1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos nx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \, dx\right)$$

The first term on the right equals $2\pi a_0$. Integration shows that all the other integrals are 0. Hence, division by 2π gives (4a).

We prove (4b), multiplying (3) on both sides by $\cos mx$ with any fixed positive integer m and integrating from $-\pi\ to\ \pi$, we have

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \int_{-\pi}^{\pi}\left[a_o + \sum_{n-1}^{\infty}(a_n \cos nx + b_n \sin nx\right] \cos mx \, dx$$

We now integrate term by term. Then on the right we obtain an integral of

$a_o \cos mx$, which is 0; an integral of $a_n \cos nx \cos mx$, which is $a_m \pi$ for $n = m$ and 0 for $n \ne m$ by (5a); and an integral of $b_n \sin nx \cos mx$, which is 0 for all $n\ and\ m$ by (5c). Hence the right side of (7) equals $a_m \pi$. division by π gives (4b) $(with\ m\ instead\ n)$.

We finally prove (4c); multiply (3) on both sides by $\sin mx$ with any fixed positive integer m and integrating from $-\pi \; to \; \pi$, we get

$$\int_{-\pi}^{\pi} f(x)\sin mx \; dx = \int_{-\pi}^{\pi}\left[a_o + \sum_{n-1}^{\infty}(a_n \cos nx + \; b_n \; \sin nx\right]\sin mx \; dx$$

Integrating term by term, we obtain on the right on integral of $a_o \sin mx, \; which \; is \; 0$; an integral of $a_n \cos nx \sin mx, which \; is \; b_m\pi$ if $n = m \; and \; 0 \; if \; n \neq m$, by (5c). This implies (4c) (with n denoted by m). This completes the proof of the Euler formulas (4) for the Fourier coefficient.

4.0 Conclusion

The derivation of the coefficients of the Fourier series would have been so rigorous and cumbersome, but with the orthogonality of the trignometric system all these were avoided, making the derivation quite simple and straight forward. This is in line with Leibniz result, that the values of various series with constant terms can be obtained by evaluating Fourier series at specific points [4].

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