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Application of the Wiener Hopf Technique on the Derivation of a Closed Form Solution of a Boundary Value Problem

George N. Emenogu

Department of mathematics
Michael Okpara University of Agriculture Umudike
Abia State,Nigeria.

Abstract

One of the difficulties involved in solving partial differential equations by transform methods is finding a general transform that applies over the entire special domain. The Wiener-Hopf technique avoids this problem by defining the transforms over certain regions and uses complex analysis to piece together the complete solution. An important aspect of this technique is the process of factorization of several functions into sums, product and quotients of two parts, one part is analytic in some lower half plane while the other is analytic in some upper half plane.

Keywords: Deterministic Finite State Automata, Malicious Signatures database and Model checking.

1.0 Introduction

Solutions of boundary value problems by transform methods has been carried out by many researchers. In this paper, we give a detailed step on the derivation of the closed form solution of boundary value problem using the Wiener Hopf technique. We consider the stress fields in an elastic homogenous isotropic material occupying the region expressed in polar coordinates

$$(r,\theta,z),\ -\infty < z < \infty, x = r\cos\theta, y = r\sin\theta, -\pi/2 \le \theta \le \pi/2 \qquad (1.1)$$

with a rigid line in-homogeneity embedded in the material in the region $y = 0\ and\ 0 \le x \le a$. The material is subjected to anti-plane shear loading by a pair of concentrated loads$T$ and Q applied at distances $landh$ from the origin. The mathematical model of the problem is a boundary value problem and it is solved using Mellin transform and the Wiener-Hopf technique.

Under the given special loading condition, the governing field equations of linear elasticity reduce to the following Laplace equation

$$W_{rr} + \frac{1}{r}W_r + \frac{1}{r^2}W_{\theta\theta} = 0, r \ge 0, -\pi/2 \le \theta \le \pi/2 \qquad (1.2)$$

$$W(r,0) = 0,\ 0 \le r \le a \qquad (1.3)$$

The non-vanishing polar stresses are

$$\sigma_{\theta z} = \frac{\mu}{r}\frac{\partial w}{\partial\theta}(r,\theta) and \sigma_{rz} = \mu\frac{\partial w}{\partial r}(r,\theta) \qquad (1.4)$$

2.0 Basic Equations and their Transformation

The boundary conditions are

$$\sigma_{\theta z}\left(r,\frac{\pi}{2}\right) = T\sigma(r-l)\ ,\ \sigma_{\theta z}\left(r,-\frac{\pi}{2}\right) = Q\sigma(r-h) \qquad (2.1)$$

Here σ is the Dirac's delta function.

The asymptotic behaviors of the stresses are

$$\sigma_{\theta z};\sigma_{rz} = \begin{cases} 0(r^{-\lambda}); 0 < \lambda < \frac{1}{2} asr \to 0 & (2.2) \\ 0(r^{-1}) asr \to \infty & (2.3) \\ 0(r-a)^{-\frac{1}{2}} asr \to a and \theta \to \infty & (2.4) \end{cases}$$

Corresponding author: ***George N. Emenogu,*** E-mail: gnemenogu@hotmail.com, Tel.: +2348076829048

The continuity conditions are

$$W(r,0^+) = W(r,0^-) = 0 \qquad 0 \le r \le a \qquad (2.5)$$

$$W(r,0^+) = W(r,0^-) = 0 \qquad r \ge a \qquad (2.6)$$

$$\sigma_{\theta z}(r,0^+) = \sigma_{\theta z}(r,0^-) = 0 \qquad r \ge a \qquad (2.7)$$

Using Mellin transform defined by

$$W(s,\theta) = \int_0^\infty W(r,\theta) r^{s-1} dr \qquad (2.8)$$

By transforming equation (1.2) and the boundary conditions in (2.1) we have

$$\int_0^\infty (r^2 W_{rr}) r^{s-1} dr + \int_0^\infty (r W_r) r^{s-1} dr + \int_0^\infty W_{\theta\theta} r^{s-1} dr = 0$$

Integrating by parts and making use of (2.2) and (2.3) we obtain

$$\frac{d^2\widehat{W}}{d\theta^2} + s^2 \widehat{W} = 0 \qquad for \lambda - 1 < Res < 0: \quad -{}^{\pi}/_{2} \le \theta \le {}^{\pi}/_{2} \qquad (2.9)$$

Transforming the boundary conditions, gives

$$\int_0^\infty W(r,\theta) r^{s-1} dr = \int_0^a W(r,\theta) r^{s-1} dr + \int_0^\infty W(r,\theta) r^{s-1} dr \qquad (2.10)$$

The first term on the RHS of (2.10) is zero

Let

$$r = a\tau, \ dr = a d\tau$$

When

$$r = a, \quad \tau = 1 \ and \ as \ r \to \infty, \tau \to \infty\,.$$

Hence

$$\int_0^\infty W(r,\theta) r^{s-1} dr = \int_1^\infty W(a\tau,0) a\tau^{s-1} a d\tau = a^{s-1} a \int_1^\infty W(a\tau,0) \tau^{s-1} d\tau$$

So that we have,

$$\widehat{W}(s,0) = a^s \widehat{V}(s)$$

where

$$\widehat{V}(s) = \int_1^\infty W(a\tau,0) \tau^{s-1} d\tau \qquad (2.11)$$

and

$$\frac{1}{u} r \sigma_{\theta z} = \frac{\partial w}{\partial \theta}(r,\theta)$$

$$\int_0^\infty W_\theta(r,\theta) r^{s-1} dr = \frac{1}{\mu}\int_0^\infty (r r \sigma_{\theta z}) r^{s-1} dr$$

This gives

$$\frac{d\widehat{w}}{d\theta}\left(s, {}^{\pi}/_{2}\right) = \frac{1}{\mu}\int_0^\infty T\sigma(r-l) r^s dr$$

$$= \frac{1}{\mu} T l^2 \qquad (2.12)$$

and

$$\frac{d\widehat{W}}{d\theta}\left(s, -{}^{\pi}/_{2}\right) = \frac{1}{\mu}\int_0^\infty Q\sigma(r-h) r^s dr = \frac{1}{\mu} Q h^2 \qquad (2.13)$$

$$\left[\frac{d\widehat{W}}{d\theta}(s,0^+) - \frac{d\widehat{W}}{d\theta}(s,0^-)\right] = \frac{1}{\mu}\int_0^\infty [\sigma_{\theta z}(r,0^+) - \sigma_{\theta z}(r,0^-)] r^s dr$$

$$= \frac{1}{\mu}\int_0^a [\sigma_{\theta z}(r,0^+) - \sigma_{\theta z}(r,0^-)] r^s dr + \frac{1}{\mu}\int_a^\infty [\sigma_{\theta z}(r,0^+) - \sigma_{\theta z}(r,0^-)] r^s dr$$

$$= \frac{1}{\mu}\int_0^a [\sigma_{\theta z}(r,0^+) - \sigma_{\theta z}(r,0^-)] r^s dr$$

Let

$$r = a\tau, \qquad dr = a d\tau$$

When

$$r = 0, \ \tau = 0 \ and when r = a, \ \tau = 1$$

$$\left[\frac{d\widehat{W}}{d\theta}(s,0^+) - \frac{d\widehat{W}}{d\theta}(s,0^-)\right] = \frac{1}{\mu}\int_0^1 [\sigma_{\theta z}(a\tau,0^+) - \sigma_{\theta z}(a\tau,0^-)] a^s \tau^s a d\tau$$

$$= a^s \frac{1}{\mu} a \int_0^1 [\sigma_{\theta z}(a\tau,0^+) - \sigma_{\theta z}(a\tau,0^-)] \tau^s \, d\tau$$

We have

$$\left[\frac{d\widehat{W}}{d\theta}(s,0^+) - \frac{d\widehat{W}}{d\theta}(s,0^-)\right] = \frac{1}{\mu}a^s\hat{u}(s) \quad (2.14)$$

where

$$\hat{u}(s) = a\int_0^1[\sigma_{\theta z}(a\tau,0^+) - \sigma_{\theta z}(a\tau,0^-)]\tau^s d\tau \quad (2.15)$$

We have the transformed equation and the boundary conditions as

$$\frac{d^2\widehat{W}}{d\theta^2} + s^2\widehat{W} = 0; \quad \lambda - 1 < Res < 0: \quad -\pi/2 \le \theta \le \pi/2 \quad (2.16)$$

$$W(s,0) = a^s\hat{V}(s) \quad (2.17)$$

$$\frac{d\widehat{W}}{d\theta}(s,\pi/2) = \frac{1}{\mu}Tl^s \quad (2.18)$$

$$\frac{d\widehat{W}}{d\theta}(s,-\pi/2) = \frac{1}{\mu}Qh^s \quad (2.19)$$

$$\frac{d\widehat{W}}{d\theta}(s,0^+) - \frac{d\widehat{W}}{d\theta}(s,0^-) = \frac{1}{\mu}a^s\hat{u}(s)$$

We consider the solution of (2.16) of the form

$$\widehat{W}(s,\theta) = \begin{cases} A_1(s)cos\theta s + B_1(s)sin\theta s & 0 \le \theta \le \pi/2 \\ A_2(s)cos\theta s + B_2(s)sin\theta s & -\pi/2 \le \theta \le 0 \end{cases}$$

where

$$A_1(s), B_1(s), A_2(s) and B_2(s)$$

are to be determined from the boundary conditions. Now using

$$\widehat{W}(s,0^+) = \widehat{W}(s,0^-) = a^s\hat{v}(s)$$

We have,

$$\frac{d\widehat{W}}{d\theta} = \begin{cases} -sA_1(s)sin\theta s + sB_1(s)cos\theta s & 0 \le \theta \le \pi/2 \\ -sA_2(s)sin\theta s + sB_2(s)cos\theta s & -\pi/2 \le \theta \le 0 \end{cases}$$

$$\frac{d\widehat{W}}{d\theta}(s,0^+) - \frac{d\widehat{W}}{d\theta}(s,0^-) = s[B_1(s) - B_2(s)]$$

$$= \frac{1}{\mu}a^s\hat{u}(s) \quad (2.20)$$

But

$$-sA_1(s)sin\,\pi/2\,s + sB_1(s)cos\,\pi/2\,s = \frac{1}{\mu}Tl^s; \quad 0 \le \theta \le \pi/2 \quad (2.21)$$

$$sA_2(s)sin\,\pi/2\,s + sB_2(s)cos\,\pi/2\,s = \frac{1}{\mu}Qh^s \quad -\pi/2 \le \theta \le 0 \quad (2.22)$$

This gives

$$\left.\begin{aligned} B_1(s) &= \frac{Tl^s + \mu s a^s\hat{v}(s)sin\pi/2 s}{\mu s cos\pi/2 s} \\ B_2(s) &= \frac{Qh^s + \mu s a^s\hat{v}(s)sin\pi/2 s}{\mu s cos\pi/2 s} \end{aligned}\right\} \quad (2.23)$$

Hence

$$s[B_1(s) - B_2(s)] = \left[\frac{Tl^s + \mu s a^s\hat{v}(s)sin\pi/2 s}{\mu s cos\pi/2 s}\right] - \left[\frac{Qh^s + \mu s a^s\hat{v}(s)sin\pi/2 s}{\mu s cos\pi/2 s}\right] \quad (2.24)$$

$$\frac{1}{\mu}a^s\hat{v}(s) = \frac{a^s}{\mu}\left[\frac{T\left(\frac{l}{a}\right)^s - Q\left(\frac{h}{a}\right)^s}{cos\pi/2 s} + \frac{2\mu s a^s\hat{v}(s)sin\pi/2 s}{cos\pi/2 s}\right] \quad (2.25)$$

Hence

$$\hat{u}(s) = \frac{T\left(\frac{l}{a}\right)^s - Q\left(\frac{h}{a}\right)^s}{cos\,\pi/2\,s} + \frac{2\mu s a^s\hat{v}(s)sin\,\pi/2\,s}{cos\,\pi/2\,s}$$

3.0 On the Fundamental Strip

We establish a common strip within which the Wiener-Hopf functions are analytic.

Let the subscript "+" denote a function that is analytic in the right half plane $Res > \lambda - 1$ and subscript "-"denote a function that is analytic in the left half plane $Res < 0$.

From (2.2) and (2.14) we see that the half known function $\hat{u}(s)$ is analytic in the right half plane $Res > \lambda - 1$. Hence we denote it by $\hat{u}_+(s)$ and from (2.3) and (2.11) it is seen that $\hat{v}(s)$ is analytic in the left plane $Res < 0$. We therefore denote it by $\hat{v}_-(s)$.

Thus

$$\hat{u}_+(s) = \frac{T\left(\frac{l}{a}\right)^s - Q\left(\frac{h}{a}\right)^s}{cos^{\pi}/_{2}s} + \frac{2\mu s a^s \hat{v}(s) sin^{\pi}/_{2}s}{cos^{\pi}/_{2}s} \tag{3.1}$$

$$\hat{u}_+(s) = \frac{2\mu s sin^{\pi}/_{2}s}{cos^{\pi}/_{2}s}\left[\hat{v}_-(s) + \frac{\frac{T}{2\mu}\left(\frac{l}{a}\right)^s - \frac{Q}{2\mu}\left(\frac{h}{a}\right)^s}{s cos^{\pi}/_{2}s}\right] \tag{3.2}$$

$$\hat{u}_+(s) = \frac{2\mu s sin^{\pi}/_{2}s}{cos^{\pi}/_{2}s}\left[\hat{v}_-(s) + \frac{\frac{T}{2\mu}\left(\frac{l}{a}\right)^s - \frac{Q}{2\mu}\left(\frac{h}{a}\right)^s}{s cos^{\pi}/_{2}(s-1)}\right] \tag{3.3}$$

Consequently, we have

$$\hat{u}_+(s) = \frac{2\mu s sin^{\pi}/_{2}s}{cos^{\pi}/_{2}s}\left[\hat{v}_-(s) + \frac{E(s)}{s cos^{\pi}/_{2}(s-1)}\right] \tag{3.4}$$

where

$$\frac{T}{2\mu}\left(\frac{l}{a}\right)^s - \frac{Q}{2\mu}\left(\frac{h}{a}\right)^s = E(s)$$

4.0 Solution of the Wiener-Hopf equation

4.1 Decomposition of the functions

To achieve the decomposition of the trigonometric function, we introduce the infinite product theorem [1].

$$\sin\frac{\pi}{2}s = \frac{\pi}{2}s\prod_{n-1}^{\infty}\left[1 - \left(\frac{s}{2n}\right)^2\right] \tag{4.1}$$

Therefore

$$\frac{2\mu s sin^{\pi}/_{2}s}{cos^{\pi}/_{2}s} = \frac{4\mu s sin^{2\pi}/_{2}s}{sin\pi s} \tag{4.2}$$

Which leads to

$$\frac{4\mu s sin^{2\pi}/_{2}s}{sin\pi s} = \frac{N_-(s)}{N_+(s)} \tag{4.3}$$

Substituting into (3.4) we have

$$\hat{u}_+(s) = \frac{N_-(s)}{N_+(s)}\left[\hat{v}_-(s) + \frac{E(s)}{s cos^{\pi}/_{2}(s-1)}\right] \tag{4.4}$$

We obtain

$$\hat{u}_+(s)N_+(s) = N_-(s)\hat{v}_-(s) + \frac{N_-(s)E(s)}{s cos^{\pi}/_{2}(s-1)} \tag{4.5}$$

The mixed term in (4.5) is decomposed into sum using the Mittag-leffler's theorem [2]

$$\frac{N_-(s)}{s} - \frac{E(s)}{s cos^{\pi}/_{2}(s-1)} = M_+(s) + M_-(s) \tag{4.6}$$

From the relationship between gamma function and infinite product [3],

$$\frac{4\mu s sin^2 \pi/_{2}s}{sin\pi s} = \frac{4\mu s\left[\frac{\pi}{2}s\prod_{n-1}^{\infty}\left[1-\left(\frac{s}{2n}\right)^2\right]\right]^2}{\pi s\prod_{n-1}^{\infty}\left[1-\left(\frac{s}{2n}\right)^2\right]}$$

$$= \frac{\mu s^2\pi\left[\prod_{n-1}^{\infty}\left(1-\frac{s}{2n}\right)\left(1+\frac{s}{2n}\right)\right]^2}{\prod_{n-1}^{\infty}\left(1-\frac{s}{2n}\right)\left(1+\frac{s}{2n}\right)}$$

$$= \mu s^2\pi\left[\frac{\left[\prod_{n-1}^{\infty}\left(1-\frac{s}{2n}\right)^2\right]}{\prod_{n-1}^{\infty}\left(1-\frac{s}{n}\right)}\right]\left[\frac{\left[\prod_{n-1}^{\infty}\left(1+\frac{s}{2n}\right)^2\right]}{\prod_{n-1}^{\infty}\left(1+\frac{s}{n}\right)}\right]$$

$$N_-(s) \quad = \quad \frac{\left[\prod_{n-1}^{\infty}\left(1-\frac{s}{2n}\right)\right]^2}{\prod_{n-1}^{\infty}\left(1-\frac{s}{n}\right)} e^{xs} \tag{4.7}$$

and

$$N_+(s) \quad = \quad \frac{\prod_{n-1}^{\infty}\left(1+\frac{s}{n}\right)e^{xs}}{\left[\prod_{n-1}^{\infty}\left(1+\frac{s}{2n}\right)\right]^2} \tag{4.8}$$

where χ will be chosen so that $N_-(s)$ and $N_+(s)$ have algebraic behavior as $|s| \to \infty$

$$\frac{1}{\Gamma(s)} = se^{xs} \prod_{n-1}^{\infty}\left(1+\frac{s}{n}\right) e^{s_n} \tag{4.9}$$

$$\frac{1}{\Gamma(-s)} = -se^{xs} \prod_{n-1}^{\infty}\left(1+\frac{s}{n}\right) e^{s_n} \tag{4.10}$$

We have

$$N(s) = \frac{-4\mu s\Gamma(-s)e^{\chi^s}}{\left[\Gamma\left(\frac{s}{2}\right)\right]} \tag{4.11}$$

$$N_+(s) = \frac{s\left[\Gamma\left(\frac{s}{2}\right)\right]}{\Gamma(s)} e^{\chi^s} \tag{4.12}$$

From Mittag-Leffler's theorem

$$sec\frac{\pi}{2}(s-1) = \frac{4}{\pi}\sum_{n-1}^{\infty}\frac{(-1)^{n+1}(2n-1)}{(2n-1)^2-(s-1)^2} \tag{4.13}$$

we have

$$E(s)N(s)sec\frac{\pi}{2}(s-1) = \frac{2N(s)E(s)}{\pi}\sum_{n-1}^{\infty}\frac{1}{s-\xi_n} - \frac{1}{s+\xi_{n-1}}$$

where

$$\xi_n = 2n; \quad \xi_{n-1} - 2n - 2$$

So that

$$E(s)N(s)sec\frac{\pi}{2}(s-1) = \frac{2}{\pi}\sum_{n-1}^{\infty}(-1)^n[N(s)E(s) - N(\xi_n)E(-\xi_n)]\frac{1}{s-\xi_n}$$

$$-\frac{2}{\pi}\sum_{n-1}^{\infty}(-1)^n\frac{N(s)E(s)}{s+\xi_{n-1}} + \frac{2}{\pi}\sum_{n-1}^{\infty}(-1)^n\frac{N(-\xi_n)E(-\xi_n)}{s-\xi_n} \tag{4.14}$$

$$M_+(s) = \frac{2}{\pi}\sum_{n-1}^{\infty}(-1)^n\left[\frac{N(s)E(s)-N(-\xi_n)E(-\xi_n)}{s-\xi_n} - \frac{N(s)E(s)}{s+\xi_{n-1}}\right] \tag{4.15}$$

and

$$M_+(s) = \frac{2}{\pi}\sum_{n-1}^{\infty}(-1)^n\frac{N(-\xi_n)E(-\xi_n)}{s-\xi_n} \tag{4.16}$$

We have from the decomposition that

$$N_+(s)\hat{u}_+(s) = \hat{v}_-(s)N_-(s) + M_+(s) + M_-(s) \tag{4.17}$$

Hence by analytic continuation

$$N_+(s)\hat{u}_+(s) - M_+(s) = \hat{v}_-(s)N_-(s) + M_-(s) = C \tag{4.18}$$

Now

$$\hat{v}_-(s)N_-(s) + M_-(s) = C$$

Considering the behavior of

$$M_-(s); N_-(s) and \hat{v}_-(s) at s = 0$$

We get

$$N_-(s) = 0; \; M_-(s) \neq 0$$

Hence

$$C = M_-(0) \tag{4.19}$$

We then have

$$\hat{v}_-(s)N_-(s) + M_-(s) = M_-(0)$$

This gives

$$\hat{v}_-(s) = \frac{M(0)-M(s)}{N(s)} \tag{4.20}$$

And $\hat{u}_+(s) = \frac{M_+(0)-M_+(s)}{N_+(s)}$ (4.21)

4.2 TheMellin Transform Formula

We have for

$$0 \le \theta \le {}^{\pi}/_{2}$$

$$\widehat{W}(s,\theta) = \frac{a^s \hat{v}(s) Cos\left(\frac{\pi}{2}-\theta\right)s}{cos\frac{\pi}{2}s} + \frac{Tl^s sin\theta s}{\mu s cos\frac{\pi}{2}s} \quad (4.22)$$

$$= a^s\left[\frac{T\left(\frac{l}{a}\right)^s Sin\theta s}{\mu s cos\frac{\pi}{2}s} + \left(\frac{M_-(0)-M_-(s)}{N_-(s)}\right)\frac{Cos\left[\frac{\pi}{2}-\theta\right]s}{cos\frac{\pi}{2}s}\right] \quad (4.23)$$

And for ${}^{\pi}/_{2} \le \theta \le 0$; we have

$$\widehat{W}(s,\theta) = a^s\left[\frac{Q\left(\frac{h}{a}\right)^s Sin\theta s}{\mu s cos\frac{\pi}{2}s} + \left(\frac{M_-(0)-M_-(s)}{N_-(s)}\right)\frac{Cos\left[\frac{\pi}{2}+\theta\right]s}{cos\frac{\pi}{2}s}\right] \quad (4.24)$$

The inversion integral gives the displacement sought for as

$$W(r,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widehat{W}(s,\theta)\, r^{-s} ds \quad (4.25)$$

Hence for $0 \le r \le a;\ 0 \le \theta \le \frac{\pi}{2}$

$$W(r,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\frac{T\left(\frac{l}{a}\right)^s Sin\theta s}{\mu s cos\frac{\pi}{2}s} + \left(\frac{M_-(0)-M_-(s)}{N_-(s)}\right)\frac{Cos\left[\frac{\pi}{2}-\theta\right]s}{cos\frac{\pi}{2}s}\right]\left(\frac{r}{a}\right)^s ds \quad (4.26)$$

And for $0 \le r \le a;\ -\frac{\pi}{2} \le \theta \le 0$

$$W(r,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\frac{Q\left(\frac{h}{a}\right)^s Sin\theta s}{\mu s cos\frac{\pi}{2}s} + \left(\frac{M_-(0)-M_-(s)}{N_-(s)}\right)\frac{Cos\left[\frac{\pi}{2}+\theta\right]s}{cos\frac{\pi}{2}s}\right]\left(\frac{r}{a}\right)^s ds \quad (4.27)$$

For $0 > Resand;\ \lambda - 1 < Res < 0,\ 0 < \lambda < \frac{1}{2}$

5.0 The inversion integral

To evaluate the inversion integral (4.26) and (4.27).

The singularities of $cos\frac{\pi}{2}s$ are all simple and are located $s \pm (2n-1)$ for all $n\epsilon N$.c is greater than the real part of any of the singularities. Since $0 \le r \le a$, by Jordan's Lemma [4], we have the contour closed on the left half plane $Res < 0$.

We use the residue theorem [5] to obtain a closed form solution of the displacement as

For $0 \le r \le a;\ 0 \le \theta \le \frac{\pi}{2}$

$$W(r,\theta) = \frac{2}{\mu\pi}\sum_{n=1}^{\infty}(-1)^n\left[(-1)\frac{T\left(\frac{l}{a}\right)^{1-2n} Sin(1-2n)\theta}{1-2n} + \mu\left(\frac{M_-(0)-M_-(1-2n)}{N_-(1-2n)}\right)Cos\left[\frac{\pi}{2}-\theta\right](1-2n)\right]\left(\frac{r}{a}\right)^{2n-1} \quad (5.1)$$

For $0 \le r \le a;\ -\frac{\pi}{2} \le \theta \le 0$

$$W(r,\theta) = \frac{2}{\mu\pi}\sum_{n=1}^{\infty}(-1)^n\left[(-1)\frac{Q\left(\frac{h}{a}\right)^{1-2n} Sin(1-2n)\theta}{1-2n} + \mu\left(\frac{M_-(0)-M_-(1-2n)}{N_-(1-2n)}\right)Cos\left[\frac{\pi}{2}+\theta\right](1-2n)\right]\left(\frac{r}{a}\right)^{2n-1} \quad (5.2)$$

6.0 Conclusion

With ingenuity in complex variables, we are able to transform the problem to a boundary value problem and then use the Wiener Hopf technique to get the desired solution to the problem. From here, it is now easy to get the stress field at the in homogeneity tips and the stress intensity factor.

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