Journal of the Nigerian Association of Mathematical Physics
***Volume* 24 (July, 2013), pp 23 – 30**
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A Fuzzy Computational Approach for Determining Activity Criticality for Project Network with Trapezoidal Activity Durations

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Abstract

In this paper, we propose a new technique for determining activity criticality in a project network with activity durations modeled as trapezoidal fuzzy numbers. The new method combines a recursive algorithm with a ranking index method based on the centre of gravity of fuzzy sets for obtaining event as well as activity times. The proposed method is demonstrated using numerical examples and is shown to be effective in determining activity criticality in project scheduling when activity durations are uncertain.

Keywords: fuzzy, ranking, trapezoidal, project network

1.0 Introduction

Traditionallythe project manager usually desires to have a foreknowledge of activities which are key to meeting planned schedules. Once he has an idea of the critical activities, he monitors them closely to ensure that they are executed as planned. Traditional methods used in project scheduling include CPM and PERT. CPM assumes that activity durations are deterministic in nature, allowing the use of the so called crisp (deterministic) activity times, In practice ,the use of crisp activity durations are unrealistic.[1]. PERT uses three time estimates to determine the expected activity duration. A major shortcoming of using PERT is that the three times estimates must follow a beta distribution, necessitating the use of elaborate statistical procedures.

In reality, the project manager estimates activity durations with very vague statements such as “the activity will take approximately 5 days” or “the activity can be executed between 6 and 7 days. This type of statement of activity time durations does not lend itself to traditional methods of project scheduling such as CPM and PERT. Fuzzy set theory has proven to be an effective way of handling such vague information [2].The decision maker only requires expressing the project duration as a fuzzy set whose members have varying degree of membership from 0 to 1. For example, if a decision maker speculates that an activity will take approximately 6 days, then he can express the fuzzy activity duration as 5, 6, 7. In the fuzzy set representation of the activity duration, 6 has a membership function of 1 while the other two values have membership functions equal to 0. This way, the decision maker has been able to express both his optimism and pessimism in specifying the activity duration.

A number of researchers have used different methods for determining criticality of activities in a project. Shanker et al [3] proposed a metric distance ranking method for fuzzy numbers to a critical path method for fuzzy project network, where the duration time of each activity in a fuzzy project network is represented by a trapezoidal fuzzy number. A numerical example was provided to explain the proposed procedure in detail. Possibility of meeting a fuzzy project in a specified time is calculated for different projects having different number of activities using fuzzy critical path method based on signed distance ranking of fuzzy numbers. Shanker et al [4] presented an analytical method for measuring the criticality in a fuzzy project network, where the duration of each activity is represented by a trapezoidal fuzzy number. They used a new defuzzification formula for trapezoidal fuzzy number and applied it to calculate the float time (slack time) for each activity in the fuzzy project network to find the critical path. Soltani and Haji [5] used a modified backward pass based on a linear programming approach which removes infeasible solutions which can result in a backward pass to solve a project scheduling problem in a fuzzy environment. Mikaeilvand et al [6] proposed a new method based on centre of mass for ranking fuzzy numbers. They presented numerical examples to illustrate the proposed method and compared with other ranking methods.

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Chanas and Zielinski [1] developed a methodology for determining criticality for a path without using generalized arithmetic operations on fuzzy numbers.

In this paper, we propose a new method for determining activity criticality in a fuzzy project network. We apply a modification of the method developed by Soltani and Haji [5] to determine the latest start times of the activities in the network and then apply a ranking method to obtain event latest and earliest times. The present method is applied to a hypothetical problem and the results compared with results compared with those obtained using crisp CPM.

2.0 Fuzzy Arithmetic

Figure 1 shows a trapezoidal fuzzy number

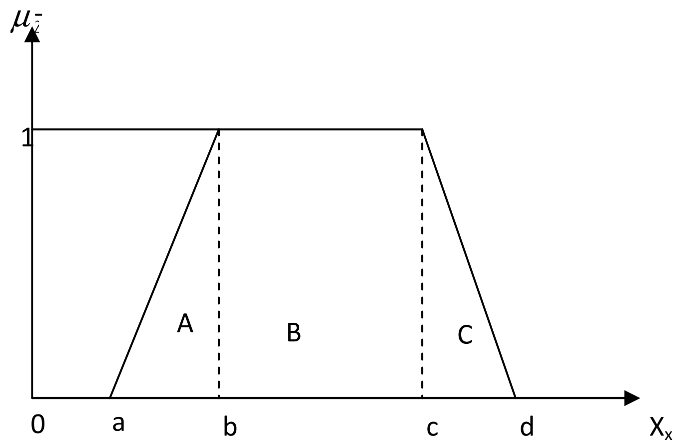

Figure 1: A trapezoidal Fuzzy Number

Let $A=(a_1,b_1,c_1,d_1)$ and $B=(a_2,b_2,c_2,d_2)$ be two flat trapezoidal fuzzy numbers. The basic fuzzy arithmetic operations namely, fuzzy addition, fuzzy subtraction and fuzzy multiplication are:

$$A\oplus B=(a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2) \quad (1)$$

$$A\Theta B=(a_1-d_2,b_1-c_2,c_1-b_2,d_1-a_2) \quad (2)$$

$$A\otimes B=(a_1a_2,b_1b_2,c_1c_2,d_1d_2) \quad (3)$$

The ranking of fuzzy numbers is an important issue in project scheduling. In forward pass computations to determine the event earliest times, we use ranking to determine the event earliest time. Cheng [7] developed a distance index method for fuzzy number comparison based on the calculation of a centroid point (x_G,y_G) to obtain the distance index. The centroid point (x_G,y_G) for a trapezoidal fuzzy number A can be defined by

$$x_G=\frac{d_1^2+c_1^2-a_1^2-b_1^2+c_1d_1-a_1b_1}{3(c_1+d_1-a_1-b_1)} \quad (4)$$

$$y_G=\frac{a_1+2b_1+2c_1+d_1}{3(a_1+b_1+c_1+d_1)} \quad (5)$$

The ranking index $R(A)$ can be obtained by the expression

$$R(A)=\sqrt{x_G{}^2+y_G{}^2} \quad (6)$$

Given any two fuzzy sets A and B whose ranking indices are $R(A)$ and $R(B)$, the comparison of the fuzzy numbers has the following properties

$$if\ \ R(A) > R(B)\ \text{then, A} > B \tag{7}$$

$$if\ \ R(A) < R(B)\ \text{then, A} < B \tag{8}$$

$$if\ \ R(A) = R(B)\ \text{then, A} = B \tag{9}$$

3.0 Fuzzy Forward Pass

Consider a fuzzy project network with n nodes. Using the activity on arrow (AOP) convention, each activity has a starting node i and an ending node j. Consequently, an activity can be denoted as A_{ij}. The nodes in the project are numbered from 1 to n, where 1 is the starting node and n is the ending node. The fuzzy event earliest and latest time of a node i are denoted by E_i and L_i respectively.

The fuzzy event earliest time of the starting node $(i=1)$ is given by equation (10).

$$E_i = (0,0,0,0) \tag{10}$$

For a node j which has a number of predecessor nodes i, we compute $\tilde{E}_i \oplus \tilde{t}_{ij}\ \forall\ i \in p(j)$

We then find the ranking index of the various values using the method proposed by Cheng [7] The $\tilde{E}_i \oplus \tilde{t}_{ij}$ fuzzy set whose ranking index is greatest becomes the fuzzy event earliest time of the node under inspection as shown in equation (11)

$$\tilde{E}_j = \left(\tilde{E}_j^1, \tilde{E}_j^2, \tilde{E}_j^3, \tilde{E}_j^4\right) = \max_{i \in p(j)}\left[R\left(\tilde{E}_i \oplus \tilde{t}_{ij}\right)\right] \qquad i \in p(j) \neq \phi \tag{11}$$

In equation (11), $i \in p(j)$ denote the set of nodes i which are predecessors to node j. For the set of all activities which have an ending node j being successors to a particular starting node i, the fuzzy earliest start time of all the activities is given by Ei. The fuzzy earliest finish time of an activity A_{ij} can be computed after the fuzzy event earliest time of node i Ei has been computed by using the expression

$$E_i \oplus t_{ij} \tag{12}$$

4.0 Fuzzy Backward Pass

In the fuzzy backward pass, we determine the fuzzy event latest time $\tilde{L}_I$, the fuzzy activity latest finish time $\tilde{L}F_{ij}$ as well as the fuzzy activity latest start time $\tilde{L}S_{ij}$. The fuzzy event latest time of the ending node (n) in the project is equal to the fuzzy event earliest time of the ending node $(E_n = L_n)$. In this paper, we propose a new technique for determining the latest event time L_i for all nodes $i < n$. In particular, we employ a modified backward pass based on a recursive algorithm to find the latest start LS_{ij} for all activities. The recursive algorithm is given as

$$\begin{aligned}
\tilde{L}S_{ij} &= \left(LS_{ij}^1, LS_{ij}^2, LS_{ij}^3, LS_{ij}^4\right) \\
LS_{ij}^4 &= \max\left[0, \left(L_j^4 - t_{ij}^4\right)\right] \\
LS_{ij}^3 &= \max\left[0,\ \min\left(LS_{ij}^4,\ \left(L_j^3 - t_{ij}^3\right)\right)\right] \\
LS_{ij}^2 &= \max\left[0,\ \min\left(LS_{ij}^3,\ \left(L_j^2 - t_{ij}^2\right)\right)\right] \\
LS_{ij}^1 &= \max\left[0,\ \min\left(LS_{ij}^2,\ \left(L_j^1 - t_{ij}^1\right)\right)\right]
\end{aligned} \tag{13}$$

After computing the fuzzy activity latest start time LS_{ij} for all activities which originate from node i, the event latest time of node i is obtained by finding the ranking indices of the respective LS_{ij} of all activities which originate from node i, and setting the LS_{ij} value with the minimum ranking index as L_i.The fuzzy activity latest finish time LF_{ij} is equal to L_j

5.0 Fuzzy Slack (Total Float) Time

The critical activities are determined by computing the slack time of the activities in the network. The activities whose fuzzy slack time is equal to (0,0,0,0) are critical. The fuzzy total float of an activity is computed using the expression

$$TF_{ij} = LF_{ij}\Theta\left(ES_{ij} \oplus t_{ij}\right) \tag{14}$$

In fuzzy network, it is necessary sometimes to convert fuzzy total float of activities to crisp values. In order to achieve this we find the centroid of the fuzzy set. Consider a fuzzy set $A=\left(a_1,b_1,c_1,d_1\right)$. To defuzzify the fuzzy set, we use the expression

$$Centroid\ \left(A\right)=\frac{a_1+b_1+c_1+d_1}{4} \tag{15}$$

6.0 Numerical Example 1

To demonstrate the concept and test the performance of the present method, a simple case example was adopted from Udosen [8]. It consists of 8 activities. The crisp activity durations reported by [8] were presented as trapezoidal fuzzy sets such that the defuzzified value (crisp) obtained using the center of gravity defuzzifier corresponds to the crispvalues. The precedence relations as well as the trapezoidal fuzzy activity durations are presented in Table 1

Table 1: Precedence relations and trapezoidal fuzzy representation of activity durations

Activity	1-2	1-3	1-4	2-3	2-4	2-5	3-5	4-5
Duration(days)	2,4,6,8	1,3,5,7	2,4,8,10	3.5.9,11	2,5,11,14	4,7,13,16	6.8.12.14	5,8,14,17

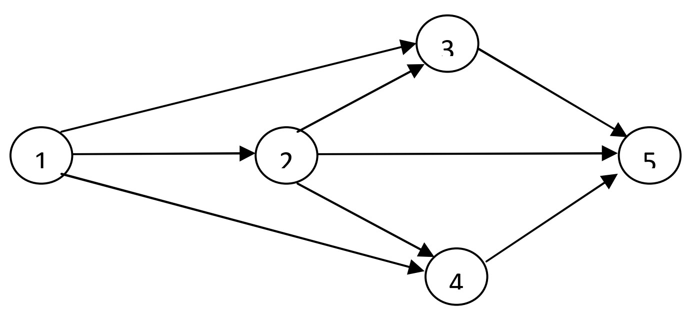

Figure 2: Network diagram for numerical example

7.0 Fuzzy Forward Pass

Step 1

The fuzzy earliest time of node 1 $\left(E_1\right)$ is the project start time equal to $\left(0,0,0,0\right)$

Step 2

The fuzzy earliest time of node 2 $\left(E_2\right)$ is computed by $E_1 \oplus t_{12}$ which gives $\left(2,4,6,8\right)$.The earliest start time of activity 1-2 $\left(ES_{12}\right)$ is equal to E_1. The fuzzy earliest finish time of activity 1-2 $\left(EF_{12}\right)$ is equal to $E_1 \oplus t_{12}$ which is equal to E_2

Step 3

To compute the fuzzy earliest time of node 3, we observe that node 3 has two predecessor nodes, namely node 1 and node 2. Therefore we calculate $E_1 \oplus t_{13}$ and $E_2 \oplus t_{23}$

$E_1 \oplus t_{13} = \left(1,3,5,7\right)$ and $E_2 \oplus t_{23} = \left(5,9,15,19\right)$

The fuzzy earliest time of node 3 can be computed by finding the ranking indices of the two fuzzy sets namely $\left(1,3,5,7\right)$

and $(5,9,15,19)$. The ranking index of fuzzy et $(1,3,5,7)$ is 1.2474 while the ranking index for $(5,9,15,19)$ is 2.9692. Since the ranking index of (5,9,15,19) is greater than that of (1,3,5,7), the earliest event time of node 3, E_3 is (5,9,15,19). The fuzzy earliest start time of activity 2-3 (ES_{23}) is equal to E_2. The fuzzy earliest finish time of activity 2-3, EF_{23} is equal to $E_2 \oplus t_{23}$ which is equal to
(5, 9, 15, 19)

Step 4

To compute the earliest event time of node 4, we note that node 4 has two predecessor nodes. They include node 1 and node 2. We therefore compute $E_1 \oplus t_{14}$ and $E_2 \oplus t_{24}$

$E_1 \oplus t_{14} = (2,4,8,10)$ and $E_2 \oplus t_{24} = (4,9,17,22)$

The ranking index of $E_1 \oplus t_{14}$ is equal to 1.7857 while the ranking index for $E_2 \oplus t_{24}$ is equal to 3.7478.Therefore the fuzzy earliest event time of node 4 is equal to $(4,9,17,22)$.The fuzzy earliest start time of activity 2-4 (ES_{24}) is equal to E_2 while ES_{14} is equal to E_1 obtained in step 1. EF_{24} and EF_{14} are computed by $E_2 \oplus t_{24}$ and $E_1 \oplus t_{14}$ which are equal to $(2,4,8,10)$ and $(4,9,17,22)$ respectively.

Step 5

To compute E_5, we first calculate the values of $E_3 \oplus t_{35}$, $E_2 \oplus t_{25}$ and $E_4 \oplus t_{45}$ and then determine their ranking indices.

$$E_3 \oplus t_{35} = (11,17,27,33)$$
$$E_2 \oplus t_{25} = (6,11,19,24)$$
$$E_4 \oplus t_{45} = (9,17,31,39)$$

The ranking indices of $E_3 \oplus t_{35}$, $E_2 \oplus t_{25}$ and $E_4 \oplus t_{45}$ equal 4.7830, 3.8193 and 6.3811 respectively Therefore E_5 is equal to $(9,17,31,39)$ since $E_4 \oplus t_{45}$ has the highest ranking index. The earliest finish times for activities 3-5,2-5 and 4-5 namely EF_{35}, EF_{25} and EF_{45} are equal to
$E_3 \oplus t_{35}$, $E_2 \oplus t_{25}$ and $E_4 \oplus t_{45}$ respectively.

8.0 Fuzzy Backward Pass

Step 1

The fuzzy latest time of event 5, L_5 is equal to $(9,17,31,39)$. This is because node 5 is the end node of the project and
$E_5 = L_5$

Step 2

The latest start time of activity 3-5, LS_{35} is obtained by using the recursive algorithm:

$$LS_{35}^4 = \max\left(0,\left(L_5^4 - t_{35}^4\right)\right)$$
$$= \max\left(0,(39-14)\right) = 25$$
$$LS_{35}^3 = \max\left(0,\min\left(LS_{35}^4,\left(L_5^3 - t_{35}^3\right)\right)\right)$$
$$= \max\left(0,\min\left(25,(31-12)\right)\right) = 19$$

$$LS_{35}^{2} = \max\left(0, \min\left(LS_{35}^{3}, \left(L_5^2 - t_{35}^2\right)\right)\right)$$

$$= \max\left(0, \min\left(19, \left(17-8\right)\right)\right) = 9$$

$$LS_{35}^{1} = \max\left(0, \min\left(LS_{35}^{2}\left(L_5^1 - t_{35}^1\right)\right)\right)$$

$$= \max\left(0, \min\left(9, \left(9-6\right)\right)\right) = 3$$

The fuzzy latest start time of activity 3-5, LS_{35} is equal to $(3,9,19,25)$. Since node 5 is the only successor node to node 3, the fuzzy latest time of event 3, L_3 is equal to $(3,9,19,25)$.

Step 3

In this step, we compute the latest start time of activity 4-5, LS_{45} and the fuzzy latest time of node 4, L_4.

$$LS_{45}^{4} = \max\left(0, \left(L_5^4 - t_{45}^4\right)\right)$$

$$= \max\left(0, \left(39-17\right)\right) = 22$$

$$LS_{45}^{3} = \max\left(0, \min\left(LS_{45}^{4}, \left(L_5^3 - t_{45}^3\right)\right)\right)$$

$$= \max\left(0, \min\left(22, \left(31-14\right)\right)\right) = 17$$

$$LS_{45}^{2} = \max\left(0, \min\left(LS_{45}^{3}, \left(L_5^2 - t_{45}^2\right)\right)\right)$$

$$= \max\left(0, \min\left(17, \left(17-8\right)\right)\right) = 9$$

$$LS_{45}^{1} = \max\left(0, \min\left(LS_{45}^{2}, \left(L_5^1 - t_{45}^1\right)\right)\right)$$

$$= \max\left(0, \min\left(9, \left(9-5\right)\right)\right) = 4$$

The latest start time of activity 4-5 is therefore equal to $(4,9,17,22)$. The latest time of event 4, L_4 is equal to $(4,9,17,22)$.

Step 4

In this step, we compute the fuzzy event latest time (E_2) of node 2. In order to achieve this, we first compute LS_{24}, LS_{25} and LS_{23} by using the recursive algorithm and then calculate their ranking indices. The particular latest start value with the least ranking index is selected as E_2. Following the procedure in steps 2 and 3, we obtain the following

$$LS_{23} = (0,4,10,14)$$

$$LS_{25} = (5,18,18,23)$$

$$LS_{24} = (2,4,6,8)$$

The ranking indices of LS_{23}, LS_{25} and LS_{24} are 2.7383, 3.7857 and 1.2783 respectively. Since LS_{24} has the least ranking index, E_2 becomes $(2,4,6,8)$

Step 5

In the step, we determine E_1, LS_{14}, LS_{12} and LS_{13}. Following the procedure in step 4, we determine LS_{12}, LS_{13} and LS_{14} to be $(0,0,0,0)$, $(2,6,14,18)$ and $(2,5,9,12)$. The ranking indices of LS_{12} LS_{13} and LS_{14} are 0, 3.3706 and 2.0616 respectively. Therefore, E_1 equal $(0,0,0,0)$. The earliest start, earliest finish, latest start, latest finish and total float times of the activities in the network are shown in Table 2. The total float of the activities was computed using equation (14).

Equation (14) contains two fuzzy arithmetic operators namely, fuzzy addition and subtraction and was manipulated using Equations(1)and(2)

Table 2: The fuzzy earliest start, fuzzy earliest finish, fuzzy latest start, fuzzy latest finish and fuzzy total float of activities in the project network.

Activity	Durations	ES	LS	EF	LF	TF
1-2	2,4,6,8	0,0,0,0	0,0,0,0	2,4,6,8	2,4,6,8	0,0,0,0
1-3	1,3,5,7	0,0,0,0	2,6,14,18	1,3,5,7	3,9,19,25	2,6,14,18
1-4	2,4,8,10	0,0,0,0	2,5,9,12	2,4,8,10	4,9,17,22	2,5,9,12
2-3	3,5,9,11	2,4,6,8	0,4,10,14	5,9,15,19	3,9,19,25	0,0,4,6
2-4	2,5,11,14	2,4,6,8	2,4,6,8	4,9,17,22	4,9,17,22	0,0,0,0
2-5	4,7,13,16	2,4,6,8	5,10,18,23	6,11,19,24	9,17,31,39	3,6,12,15
3-5	6,8,12,14	5,9,15,19	3,9,19,25	11,17,27,33	9,17,31,39	0,0,4,6
4-5	5,8,14,17	3,9,17,22	4,9,17,22	9,17,31,39	9,17,31,39	0,0,0,0

9.0 Numerical Example 2

Figure 3 shows the network presentation of a fuzzy project network [4]

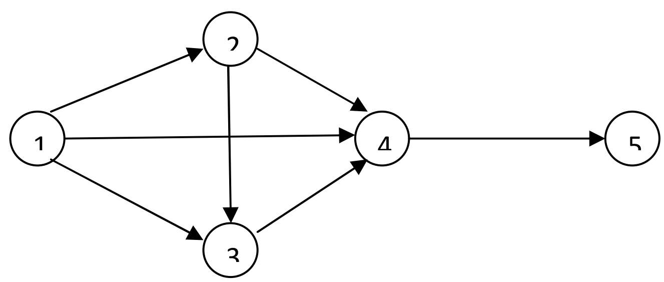

Figure 3: Precedence diagram of network

Table 3 shows the results obtained by Shanker et al [4], who applied an analytical method to the fuzzy project network and the results obtained using the present method.

Table 3: Results Obtained for Numerical example 2

Activity	Fuzzy activity time	Slack time Shanker et al [4]	Defuzzified Slack time Shanker at al[4]	Slack time Present method	Defuzzified Slack time Present method
1-2	(10,15,15,20)	(-160,-60,60,160)	0	(0.0.0.0)	0
1-3	(30, 40,40,50)	(-130,-35,75,170)	20	(30,25,15,10)	20
2-3	(30,40,50,60)	(-160,-60,60,160)	0	(0,0,0,0)	0
1-4	(15,20,25,30)	(-110.-20,95,185)	37.5	(50,40,35,25)	37.5
2-5	(60,100,150,180)	(-100,-10,100,190)	45	(60,50,40,30)	45
3-5	(60,100,150,180)	(-160,-60,60,160)	0	(0,0,0,0)	0
4-5	(60,100,150,180)	(-110.-20,95,185)	37.5	(50,40,35,25)	37.5

10 Discussion

The result of the fuzzy critical path analysis to example 1 is shown in Table 2. The result shows that three activities namely 1-2, 2-4 and 3-5 are critical since their fuzzy total float values are $(0,0,0,0)$. The non critical activities are 1-3, 1-4, 2-3, 2-5 and 3-5 respectively. The defuzzified total floats of the non critical activities are 10, 7, 2.5, 9 and 2.5 respectively. Comparison of the results obtained using the present fuzzy approach to that obtained when the defuzzified fuzzy durations are employed using crisp CPM show that the crisp CPM produces the same critical activities as the present approach. Table 4 shows the result obtained using the present method to the problem solved by Shanker et al [4] who solved the same problem using an analytical technique. Shanker et al [4] used a method which allows the occurrence of negative fuzzy numbers in contrast to the present method. In particular, we used a backward approach which eliminates the possibility of negative fuzzy numbers. Table 3 shows that the present method results in the same critical activities as the method used by Shanker et al [4]. The defuzzified total float obtained using the present method is identical to that obtained by [4]. The present method has been shown to be effective in determining activity criticality in fuzzy project networks.

Conclusion

A new method has been applied to project scheduling with uncertain activity durations. The method has been shown to be effective in determining activity criticality in a project network

The method produces consistent results with other methods whose solutions are available in the literature. The method has been shown to be invaluable for project planners.

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