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Properties of Practical Numbers

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Abstract

The practical number system may not be as popular as other number systems in number theory, but, it certainly cannot be ignored in modern day mathematics. Many mathematical concepts in analysis and other aspect of mathematics and physics can be simplified using the concept of practical numbers. In this work, we examine some properties of the practical number system.

Keywords: practical numbers, divisors, number systems, Egyptian fractions, perfect numbers.

1.0 Introduction

Let m be a positive number. If m is such that every integer n ε (1,m) can be expressed as a sum of distinct positive divisors of m, then m is called a practical number. Srinivasan [1]was the first person to define practical numbers. Some other authors refers to them as panarithmic numbers [2].

An example of practical number is 12. 12 is a practical number because all the numbers from 1 to 11 can be expressed as sums of divisors of 12. i.e.

$$2=1+1,\ 3=2+1,\ 4=3+1,\ 5=3+2,\ 7=6+1,\ 8=6+2,\ 9=6+3,\ 10=6+3+1,\ 11=6+3+2.$$

The sequence of practical numbers as given in [3] is
1,2,4,6,8,12,16,18,20,24,30,32,36,40,42,54,56,60,...

The practical numbers system is not as popular as other known number systems such as the complex number system, the real number system, the rational number system, the prime number system etc. However, the importance of this number system can be seeing in the examples below.

(a) The practical number system is different from the discrete systems in that, the operations of addition and multiplication must be defined for each pair of numbers in any discrete system but that is not the case with practical numbers. For example, if the largest integer is k, then the product k x k is not defined, nor is the sum k+1 defined. In pure mathematics spaces where this product and sum cannot be defined can be studied using the practical number system [2].

(b) We think it is reasonable to suspect that most important theorems of analysis can be proved in the practical number system, just as it is with the real number system.

(c) In the practical number system, there are no infinite series. All series are finite [4].

(d) We observe, the implication of the practical number system on topology and measure theory will be interesting as all sets would be closed and there would be no open sets.

(e) In practical number system, there is a lower limit on the size of positive, non-zero numbers [1].

2.0 Properties of Practical Numbers

(a) – (e) above and many more properties of practical numbers is the reason why the system of numbers cannot be ignored. Several authors dealt with some of its important properties these include:

Erdos [5] who announced that practical numbers have zero asymptotic density. Stewart [6], proved that an integer $m \geq 2$, $m = p_1^{\alpha_1},\ p_2^{\alpha_2} \cdots p_k^{\alpha_k}$,with primes

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$p_1,\ p_2 < \cdots < p_k$ and integers $\alpha_i > 1$,

Is practical if and only if $p_1 = 2$ and for $i = 2,3,...k$,

$$p_i \le \sigma\left(p_1^{\alpha_1},\ p_2^{\alpha_2} \cdots p_{i-1}^{\alpha_{i-1}}\right) + 1$$

Where $\delta(n)$ denotes the sum of the positive divisors of n.

Let $p(x)$ be the counting function of practical numbers $p(x) = \sum_{m \le x} m$, m practical, Margenstem [7] developed upper and lower bounds for the counting function as follows

$$\frac{x}{\log x}(\log\log x)^{5/3-\varepsilon} \le p(x) \le x \log x(\log\log x \log\log\log x)$$

Melfi [8] conjectured that $p(x) \sim \lambda x / \log x, with$

$\lambda \simeq 1.341$, in analogy with the asymptotic behaviour of primes.

In this work, we look at some of these properties with the aim of improving upon then.

Lemma 1: Let m be a positive integer and let $d_1 = 1 < d_2 < d_2 < \cdots < d_r = m$ be the positive divisors of m let d_h be the least divisor such that $d_h \ge \sqrt{m}$. Then $d_1 + d_2 + \cdots + d_{h-1} + 1 \le m$.

Theory 1: A positive integer m is practical if and only if every n with $1 \le n \le \alpha(m)$ is a sum of distinct positive divisors of m.

Proof: Since $\delta(m) \ge m$, if m is such that every n with $1 \le n \le \delta(m)$ is a sum of distinct divisors of m, then m is practical.

Let m be practical, i.e. every n with $1 \le n \le m$ is a sum of distinct positive divisors of m. let $d_1,..., d_{h,} ...d_r$ be as in a lemma 1. For any j satisfying $1 \le j \le h-1$ we have $d_1 + \cdots + d_j + 1 \le m$ by lemma 1. Hence $d_1 + \cdots + d_j + 1$ is a sum of distinct divisors of m, of which at least one must be $\ge d_j + 1$. It follows that $d_j + 1 \le d_1 + \cdots + d_j + 1$ whence every n with $1 \le n \le \sigma(m)$ is a sum of distinct divisors of m.

Lemma 2: If m is a practical number and n is an integer such that $1 \le n \le \delta(m) + 1$,then mn is a practical number. In particular for $1 \le n \le 2m$, mn is practical.

Theorem 2: Every practical number x can be expressed as a sum of a finite set of practical numbers.

Proof: Let n be a positive integer; if every $n \in (1, m)$ can be expressed as a sum of distinct divisors of m, then m is practical. Also, we argue that every practical number $x \in (1,\ m)$ can be expressed as sum of distinct practical numbers less than x.

Let $x^p \in (1,\ m)$ be a practical number, if $x_1,\ x_2, \cdots, x_k \in (1,\ m) < x^p$ and $x_1,\ x_2, \cdots, x_k$ are practical numbers and divisors of x^p then

$$x^p = x_1 + x_2 + ... + x_k, x_k \lhd x^p.$$

That is x^p can be expressed as sum of practical numbers within $(1, m)$.

It's important to note that sometimes some numbers may be repeated to get the desired x^p i.e.

$$x^p = x_1 + x_1 + x_2 + ...$$

$$x^p = x_1 + x_2 + x_2 + ...$$

However , where possible repetition should be avoided. A good example, is the practical number 18.

$x^p \in (1, 18) = \{1, 2, 4, 6, 8, 12, 16\}$,

Let $x^p = 12,$ we have 12 = 6 +4+2,

$x^p = 8$, we have 8 = 6+2,

$x^p = 6,$ we have 6 = 4+2,

x^p = 4, we have, 4=2+2,

x^p = 2, we have, 2=1+1.

Theorem 3: If m and $m+2$ are two practical numbers, then every even integer $2n$ with $m^2 \le 2n \le \frac{7}{2}m^2$ is a sum of two practical numbers.

Proof: Melfi [8], split up the interval $\left[\frac{m^2}{2}, \frac{7m^2}{2}\right]$ into

(i) $\left[\frac{1}{2}^{m^2}, m^2\right]$,

(ii) $[m^2, 3m^2]$,

(iii) $[3m^2, 7/2^{m^2}]$.

We split the interval into five sub intervals .

(i) $[m^2/2, m^2)$

(ii) $[m^2, 2m^2]$

(iii) $\left[2m^2, \frac{5}{2}m^2\right]$

(iv) $\left[\frac{5}{2}m^2, 3m^2\right]$

(v) $(3m^2, 7m^2/2]$

by refining the closed interval $\left[\frac{m^2}{2}, \frac{7}{2}m^2\right]$, we hope to improve upon the result obtained by Melfi[8]. In the first interval, if $m = 2$, the only even number contained in $[m^2/2, m^2)$is 2 which is a sum of two practical number $(2 = 1+1)$. Suppose $m > 2$ and let $2n \in [m^2/2, m^2)$. If $2n = \frac{m^2}{2}$ or $2n = \frac{m^2}{2} + m$, we use the decomposition

$$\frac{m^2}{2} = m\left(\frac{m}{2} - 1\right) + m, \tag{1}$$

$$m^2/2 + m = m(m/2 - 1) + 2m \tag{2}$$

Otherwise we can represent 2n as $\frac{m^2}{2} + km + 2j$ with $0 \le k < \frac{1}{2}m$, $1 \le j \le \frac{m}{2}$, $(k, j) \ne \left(0, \frac{1}{2}m\right)$ then

$$2n = \frac{m^2}{2} + km + 2j = m\left(\frac{m}{2} + k - j\right) + (m+2)j \tag{3}$$

which imply 2n is the sum of two practical numbers.

In the interval(iii), if $m=2$, the even numbers contained in the interval $\left[2m^2,\ 5/2m^2\right]$ are 8 and 10 i.e. $(10=8+2)\ and\ (8=6+2)$. Suppose $m>2$ and let 2n $\in \left[2m^2,\ 5/2m^2\right]$, we can represent 2n as

$$2m^2+km+2j=m(2m+k-j)+(m+2)j \tag{4}$$

$$\frac{5}{2}m^2+km+2j=m\left(\frac{5}{2}+k-j\right)+(m+2)j \tag{5}$$

From (4) and (5), its clear that 2n is the sum of two practical numbers.

Again in interval (iv): if $m=2$, in $\left[\frac{5}{2}n^2, 3m^2\right]$

The even number in the interval will be 10 and 12 i.e. $(10=8+2)\ and\ (12=10+2)$ which are sums of two practical numbers. Suppose $m>2$, and let $2n\in\left[\frac{5}{2}n^2,\ 3m^2\right]$, then,

$$2n=\frac{5}{2}m^2-km+2j=m(2m-k-j-1)+(m+2)\left(\frac{1}{2}m+j\right) \tag{6}$$

or

$$2n=3m^2-km+2j=m(2m-k-j-2)+(m+2)(m+j) \tag{7}$$

(6) and (7) shows 2n is the sum to two practical numbers.

In the interval (v): if $m=2$, the only even number contained in $\left(3m^2, 7m^2/2\right]$ is 14 which is the sum of two practical numbers $(14=8+6)$. Suppose $m>2$ and let $\left(3m^2, 7m^2/2\right]$. We can represent 2n as

$2n=\frac{7}{2}m^2-km+2j=m(2m-k-j-3)+(m+2)\left(\frac{3}{2}m+j\right)$ which is the sum two practical numbers.

Thus every integer $2n$ with $m^2/2\le 2n\le 7m^2/2$ is a sum of two practical number.

3.0 Egyptian Fractions and Practical Numbers

Definition: An Egyptian fraction is a sum of positive (usually) distinct unit fractions. e.g. $\frac{2}{7}=\frac{1}{4}+\frac{1}{28}$ (in this representation no unit fraction is repeated) [9].

We attempt to marry this two number systems (practical and Egyptian) with the aim of decomposing any fractions $\frac{p}{q}$ into its Egyptian form (sum of reciprocals).

Its easily seen that if p can be written as the sum of divisors of q, then $\frac{p}{q}$ can be expanded with no denominator greater than q itself, for example, if we want to expand $\frac{9}{20}$, note that 4 and 5 are divisors of 20 so that,

$$\frac{9}{20}=\frac{4+5}{20}=\frac{1}{5}+\frac{1}{4}$$

Izevbizua and Okoromi [3] provided a theorem on this relationship, which We state without proof.

Theorem 4: $\frac{m}{n}=\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_k}$ if and only it there exist positive integers M and N and divisors $D_1\cdots D_k$ of N such that $\frac{m}{N}=\frac{m}{n}\ and\ D_1+D_2+\cdots+D_k\ =0$ (modM). Also, the last condition can be replaced by

$D_1 + D_2 + \cdots + D_k = M$; and the condition $(D_1, D_2, \cdots, D_k) = 1$ may be added without affecting the validity of theorem.

Another theorem connecting Egyptian fractions and practical numbers was proved by Robinson [2].

Theorem 5: If n is a practical number and q is any number relativity prime with n, and $q < 2n$, then qn is also practical.

This theorem whose proof we have left out, shows another strong relationship between practical numbers and Egyptian fractions. We demonstrate this relationship with the aid of an example.

Using theorem 5, we can expand $\frac{5}{23}$ as follows,

$$\frac{5}{23} = \frac{5(12)}{23(12)}$$

Since 23 < 2(12),{q=23 and n=12 by theorem5} and 12 is practical, we know that 23(12) is practical(theorem5). So 5(12) can be written as the sum of distinct divisors of 23(12) i.e.,

$$\frac{5(12)}{23(12)} = \frac{46+12+2}{23(12)} = \frac{1}{6} + \frac{1}{23} + \frac{1}{138}.$$

This transition from practical numbers to Egyptian fractions and Egyptian fractions to practical numbers is of great benefit to number theory in particular and mathematics in general.

4.0 Practical Numbers and Perfect Numbers

Definition: A perfect number is a positive integer that is equal to the sum of its proper positive divisors, that is the sum of its positive divisors excluding the number itself. Equivalently, a perfect number is a number that is half the sum of all its positive divisors (including itself).

The first perfect number is 6, because 1, 2, and 3 are its proper positive divisors and 1+2+3=6. Equivalently the number 6 is equal to half the sum of all it positive divisors i.e. (1+2+3+6)/2=6.

The first four perfect numbers can be generated by $2^{p-1}(2^p - 1)$, p is prime {p = 2,3,5,7}, i.e.

$$p = 2 : 2^1(2^2 - 1) = 6,$$
$$p = 3 : 2^2(2^3 - 1) = 28,$$
$$p = 5 : 2^4(2^5 - 1) = 496,$$
$$p = 7 : 2^6(2^7 - 1) = 8128.$$

Next, we look at the relationship between practical numbers and perfect numbers.

Theorem6: Every perfect even number is also a practical number.

Proof: Let $x \in N$ be a perfect even number. Then by definition x has proper positive divisors $k_1, k_2, k_3, \ldots k_n$ such that $x = k_1 + k_2 + k_3 + \ldots + k_n$.

Also since x is perfect, then $\{k_1 + k_2 + k_3 + \ldots + k_n + x\}/2 = x$.

Note also, that $k_1, k_2, k_3, \ldots k_n$ are distinct positive divisors of x and $k_n \lhd x$.

Recall that by definition, x is practical if every $c \in (1, x)$ can be expressed as the sum of distinct positive divisors of x.

Thus, if $c_1, c_2, c_3 \ldots c_n \in (1, x)$ and

$$c_1 = k_1 + k_2,$$
$$c_2 = k_1 + k_2 + k_3,$$
$$c_3 = k_1 + k_2 + k_4,$$
$$\vdots$$
$$c_n = k_2 + k_4 + k_6 + \ldots + k_n.$$

Then x is practical.

Conclusion

The practical number system has its unique properties that makes it useful to number theory and mathematics in general. We have shown the relationship that exit between the practical numbers and some other number systems like, the Egyptian fractions and perfect numbers. We have also simplified some known theorems on practical numbers.

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