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Ergodicity Theorem For Quantum Dynamical Semigroups on von Neumann Algebras

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Abstract

Noncommutative L_p spaces technique is applied in the analysis of a quantum dynamical semigroup with a view of addressing the question of ergodicity of the extended stochastic dynamics P_t^X over a von Neumann algebra $\widetilde{\mathcal{M}}$ involving operators of the form xy^α.

Keywords: von Neumann algebras, non commutative L_p –spaces, quantum stochastic dynamics.

1.0 Introduction

The aim of quantum open system theory is to study the interaction of simple quantum system interacting with very large ones. In general the properties that one is seeking are to exhibit the dissipation of the small system in favour of the large one, to identify when this interaction gives rise to a return to equilibrium or a thermalization of the small system [1]. There are two ways of studying these systems namely, the Hamiltonian approach and the Markovian approach. In the Markovian approach, one gives up the idea of modelizing the reservoir and concentrates on the effective dynamics of the small system. This dynamics is supposed to be described by a semigroup of completely positive maps admitting a generator of the Lindblad form [2]. The aim is then to study the ergodic properties of that quantum dynamical system. In classical settings, ergodic theory deals with the study of invariant measures and their connections with the large time behaviour of dynamical systems. Ergodic theory also played an important role in the development of the algebraic approach to quantum field theory and quantum statistical mechanics, mainly in connections with analysis of symmetries [1]. An attempt to use the theory of noncommutative L_p spaces for the construction and analysis of the infinite volume dynamics for spin systems on a lattice was initiated by [3]. The extended quantum stochastic dynamics for spin system on a lattice is defined as the thermodynamic limit of the finite volume evolution P_t^{X,Λ_n}, hence the extended dynamical semigroup of the system can be written as $P_t^X = \lim_{\Lambda_n} P_t^{X,\Lambda_n}$. We are interested in the ergodicity of the extended time evolution P_t^X .

1.1 Preliminaries:

We will start with some basic relevant definitions from [4].

Definition: 1

A strongly continuous one parameter semigroup $(P_t)_{t\geq 0}$ on a Hilbert space $\mathfrak{H}$ is a family P_t of linear maps satisfying the following three conditions, $P_0 = 1$, $P_s\, P_t = P_{s+t}$ $\lim_{t\to 0} P_t u = u,$ where $u \in \mathfrak{H}$. A semigroup $(P_t)_{t\geq 0}$ on a von Neumann algebra $\mathcal{M}$ is said to be (i) a contraction semigroup if $\|P_t x\| \leq \|x\|$ $\forall\, x \in \mathcal{M}\,, t \geq 0$., (ii) uniformly continuous if $\lim_{t\to 0}\|P_t - I\| = 0,$ (iii) strongly continuous or C_0 −semigroup if $\lim_{t\to 0}\|P_t x - x\| = 0$ $\forall\ x \in \mathcal{M}.$

Definition: 2

A quantum Sub-Markov semigroup, or quantum dynamical semigroup (q.d.s) on a von Neumann algebra $\mathcal{M}$, is a one parameter family $(P_t)_{t\geq 0}$ of linear maps of $\mathcal{M}$ into itself satisfying.

a) $P_t(x) = x$ for all $x \in \mathcal{M}$.
b) Each $P_t(.)$ is completely positive.
c) $P_t(P_s) = P_{t+s}$ for all t, s$\geq$ 0.
d) $P_t(\mathrm{I}) \leq \mathrm{I}\ for\ all\ t \geq 0.$
e) For each $x \in \mathcal{M},\ the\ map\ \ t \to P_t(x)$ is σ-weakly continuous on $\mathcal{M}$
f) P_t is a normal operator on $\mathcal{M}$ for all $t \geq 0$, i.e. for every increasing net $(a_\alpha)_\alpha$ in $\mathcal{M}$ with l.u.$b\ a_\alpha = a \in \mathcal{M}$, we have l.u.$b\ P_t(a_\alpha) = P_t(a)$.
g) A quantum dynamical semigroup is called a quantum Markov semigroup if
    a. $P_t(1) = 1\ \ for\ all\ t \geq 0.$

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2.0 The algebraic setting

We are going to work within the framework of noncommutative L^p spaces arising from a von Neumann algebra. A von Neumann algebra is a *-subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathfrak{H})$ which is self-adjoint, containing the identity operator I and is closed in the weak operator topology. The weak operator topology is induced by the family of semi norms $\{p_{\xi,\eta}\}$ defined on $\mathcal{M}$ by $p_{\xi,\eta}(x) = \sum|\langle x\xi,\eta\rangle|$, with $x \in \mathcal{M}, \xi, \eta \in \mathfrak{H}$. $\mathcal{M}_+$ denote the positive elements of $\mathcal{M}$, i.e $\mathcal{M}_+ = \{x \in \mathcal{M}: x \geq 0\}$. A linear positive functional φ on $\mathcal{M}$ is called a state if $\varphi(1) = 1$. The space of all σ- weakly continuous linear functionals on a von Neumann algebra $\mathcal{M}$ is called the predual $\mathcal{M}_*$, we denote by $\mathcal{M}_{*,+}$ the positive part of $\mathcal{M}_*$. More details on von Neumann algebras will be found in references [4,5,6]. Let $\mathcal{M}$ be a von Neumann algebra and y a closed positive self adjoint operator affiliated to $\mathcal{M}$. Let p be a projection on the Hilbert space $\mathfrak{H}$ such that $py \subset yp$ and yp is a positive bounded everywhere –defined operator on $\mathfrak{H}$, we say that p is a bounding projection for y. Now let (p_n) be an increasing sequence of projections each of which is bounding for y and $\bigvee_{n=1}^{\infty} p_n = I$, we say that (p_n) is a bounding sequence for y and yp_n is a positive bounded everywhere –defined operator on $\mathfrak{H}$ [6].

Let $y_n = yp_n$ be considered as a bounded operator with spectrum by $sp(y_n)$. Let $C(sp(y_n))$ be the von Neumann algebra of all continuous real valued functions on $sp(y_n)$ and let $\mathcal{B}(\mathfrak{H})_+$ be the set of positive operators in $\mathcal{B}(\mathfrak{H})$. Applying functional calculus, we define y_n^α as $f_a(y_n)$ for real values of a. Thus $f_a(y_n) = y_n^\alpha \in \mathcal{B}(\mathfrak{H})_+$.

For a self-adjoint operator $x \in \mathcal{M}$, let $\widetilde{\mathcal{M}} = \{\tilde{x} : u.y_n^\alpha = \tilde{x},\ u \in \mathcal{M}, y_n^\alpha \in \mathcal{B}(\mathfrak{H})_+\}$ be the set of strong product of bounded operators on $\mathfrak{H}$, this is a *-algebra when endowed with the operations of product and involution defined as follows,

$$(u.y_n^\alpha)(v.y_n^\alpha) = uv.y_n^\alpha, \qquad u.v \in \mathcal{M}$$
$$(v.y_n^\alpha)^* = x^*.y_n^\alpha, \qquad x^* \in \mathcal{M}$$

For $y_n^\alpha \in \mathcal{B}(\mathfrak{H})_+$, the set $\widetilde{\mathcal{M}}$ is clearly a *- subalgebra of $\mathcal{B}(\mathfrak{H})$ and $\tilde{x}$ is a strong product of bounded operators on $\mathfrak{H}$ of the form $y_n^\alpha.x.y_n^\alpha$, that is,

$$y_n^\alpha x y_n^\alpha = (1.y_n^\alpha)(x.y_n^\alpha) = (1.x).y_n^\alpha = x.y_n^\alpha = \tilde{x} \in \widetilde{\mathcal{M}},$$

and is also self-adjoint, in the sense that

$$\tilde{x}^* = (y_n^\alpha x y_n^\alpha)^* = (x.y_n^\alpha)^* y_n^\alpha = y_n^\alpha x^* y_n^\alpha = x^*.y_n^\alpha = x.y_n^\alpha = \tilde{x}$$

since x is assumed to be self-adjoint. Thus $\widetilde{\mathcal{M}}$ is a unital weakly closed *- subalgebra of $\mathcal{B}(\mathfrak{H})$ hence a von Neumann algebra.

Now let $\mathbb{Z}^d, d \geq 1$ be the d -dimensional lattice, whose sites are occupied by spin- $\frac{1}{2}$ particles. One associates with each point $\boldsymbol{j} = (j_1, j_2, \dots . j_d) \in \mathbb{Z}^d$ a Hilbert space $\mathfrak{H}_{\{j\}}$ and with each finite subset $\Lambda \subset \mathbb{Z}^d$ the tensor product space $\mathfrak{H}_\Lambda = \bigotimes_{\boldsymbol{j}\in\Lambda} \mathfrak{H}_{\{j\}}$. The self-adjoint operators at site $\boldsymbol{j} = (j_1, j_2, \dots . j_d) \in \mathbb{Z}^d$ are elements of the point algebra $\widetilde{\mathcal{M}}_{\{\boldsymbol{j}\}}$. The von Neumann algebra $\widetilde{\mathcal{M}}_{\{\boldsymbol{j}\}}$ is isomorphic to a 2×2 matrix algebra $\mathcal{M}_2(\mathbb{C})$. The algebra of self-adjoint operators localized to a finite region $\Lambda \subset \mathbb{Z}^d$, defined by $\widetilde{\mathcal{M}}_\Lambda = \bigotimes_{\boldsymbol{j}\in\Lambda} \widetilde{\mathcal{M}}_{\{\boldsymbol{j}\}}$, is then the full matrix algebra $\mathcal{M}_{2^{|\Lambda|}}(\mathbb{C})$. Let $\mathcal{F}$ be the set of all finite subsets of $\mathbb{Z}^d$ ordered by inclusion, and let $\Lambda_1, \Lambda_2 \in \mathcal{F}$ be two disjoint finite regions, that is $\Lambda_1 \cap \Lambda_2 = \emptyset$. Then $\mathfrak{H}_{\Lambda_1 \cup \Lambda_2} = \mathfrak{H}_{\Lambda_1} \otimes \mathfrak{H}_{\Lambda_2}$, and we write $\widetilde{\mathcal{M}}_{\Lambda_1 \cup \Lambda_2} = \widetilde{\mathcal{M}}_{\Lambda_1} \otimes \widetilde{\mathcal{M}}_{\Lambda_2}$ for the matrix algebra.

$\widetilde{\mathcal{M}}_{\Lambda_1}$ is isomorphic to the matrix subalgebra $\widetilde{\mathcal{M}}_{\Lambda_1} \otimes I_{\Lambda_2}$ of $\widetilde{\mathcal{M}}_{\Lambda_1 \cup \Lambda_2}$,where I_{Λ_2} denotes the identity on $\mathfrak{H}_{\Lambda_2}$. Identifying $\widetilde{\mathcal{M}}_{\Lambda_1}$ and $\widetilde{\mathcal{M}}_{\Lambda_1} \otimes I_{\Lambda_2}$ one concludes that the algebra $(\widetilde{\mathcal{M}}_\Lambda)_{\Lambda\in\mathcal{F}}$ form an increasing family of matrix algebras, whose union $\bigcup_{\Lambda\in\mathcal{F}} \widetilde{\mathcal{M}}_\Lambda$ is a normed *-algebra, which is incomplete because $\mathbb{Z}^d$ is infinite. Considering the closure with respect to the norm topology, we have that $\overline{\bigcup_{\Lambda\in\mathcal{F}} \widetilde{\mathcal{M}}_\Lambda}^{\|\cdot\|} \equiv \mathcal{M}_0$ is a quasilocal von Neumann algebra [3] . We have the following definitions

2.1 Non Commutative L_P –Spaces:

We defined L_p -spaces over the quasi-local von Neumann algebra $\mathcal{M}_0$. The L_p –spaces of interest is the Trunov L_p –spaces [7]. The construction is as follows: Let τ be a faithful normal semifinite trace $\mathcal{M}_0$. The set of positive nonsingular self-adjoint operators with a finite trace is given by $\{\tilde{h} \in \mathcal{M}_0 : \tau|\tilde{h}| < \infty\}$, we denoted this set by $L_1(\mathcal{M}_0)$. Now for $\tilde{x} \in \mathcal{M}_0$, $\tilde{h} \in L_1(\mathcal{M}_0)$, we have from [8], the representation φ on $\mathcal{M}_0$ defined by $\varphi(\tilde{x}) = \tau(\tilde{x}.\tilde{h})$. Thus $\varphi(\tilde{x}) = \tau(\tilde{x}.\tilde{h}) = \tau(\tilde{h}.\tilde{x}) = \tau\left(\tilde{h}^{\frac{1}{2}}.\tilde{x}.\tilde{h}^{\frac{1}{2}}\right)$. This representation enables one to define for each $1 \leq p < \infty$ a norm $\|.\|_p$ on $\mathcal{M}_0$. For $\tilde{h} \in L_1(\mathcal{M}_0)$, we have the norm $\|\tilde{x}\|_p = \tau\left(\left|\tilde{h}^{\frac{1}{2p}}\tilde{x}\tilde{h}^{\frac{1}{2p}}\right|^p\right)^{\frac{1}{p}}$, the set $L_p(\mathcal{M}_0) = \{\tilde{x} \in \mathcal{M}_0 : \|\tilde{x}\|_p < \infty\}$ is a Banach space of pth-power integrable operators in $\mathcal{M}_0$. We set $L_\infty(\mathcal{M}_0) = \mathcal{M}_0$ and the predual $\mathcal{M}_* = L_1(\mathcal{M}_0)$

3.0 Ergodicity of Quantum Dynamical Semigroup on $\mathcal{M}_0$

In classical setting the dynamical system comprising of a triple namely the phase space X, a measurable space that is a set equipped with a σ field and continuous map index by time φ_t and a measure μ that is the triple (X, φ_t, μ) is called ergodic if for $f \in L^1$ one has

$E_\mu(f(x)) = \mu(f(x))$ for μ −almost all $x \in X$, where E_μ is the expectation. In this case we also say that μ is an ergodic measure for φ_t. To generalized the classical L_p −spaces technique to the quantum setting, we need a noncommutative L_p −spaces and this is realized by a von Neumann algebra. The triple (X, φ_t, μ) is then replaced with its quantum counterpart of a dynamical system, namely, the triple $(\mathcal{M}_0, P_t^X, \omega)$, where $\mathcal{M}_0$ is a von Neumann algebra, P_t^X is the finite volume stochastic dynamics, and ω is a faithful normal state. This constitutes a basis for a description of infinite quantum system. The quantum dynamical semigroup defined by $P_t^X = e^{t\mathcal{L}^X}$ is a positive, unit-preserving map on $\mathcal{M}_0$ such that the extended map is L_2 − symmetric with respect to the inner product induced by ω and contractive with respect to the $L_p(\mathcal{M}_0)$ norm. We outline formally these properties as follows ,

(i) $P_0^X = 1$
(ii) $\langle P_t^X\tilde{x}, \tilde{y}\rangle = \langle \tilde{x}, P_t^X\tilde{y}\rangle$, $\quad \tilde{x}, \tilde{y} \in L_2(\mathcal{M}_0)$
(iii) $\|P_t^X\tilde{x}\|_{L_p(\mathcal{M}_0)} \le \|\tilde{x}\|_{L_p(\mathcal{M}_0)}$, $\quad \tilde{x} \in L_p(\mathcal{M}_0)$
(iv) $\varphi(P_t^X\tilde{x}) = \varphi(\tilde{x})$, $\quad \tilde{x} \in \mathcal{M}_0$.

When we consider processes like dissipation, the generator of the dynamical semigroup P_t^X that describes such processes is the map $\mathcal{L}^X : \mathcal{M}_0 \longrightarrow \mathcal{M}_0$, defined by,

$$\mathcal{L}^X(\tilde{x}) = E_X(\tilde{x}) - \frac{1}{2}\{E_X(1), \tilde{x}\},$$

where the map $E_X : \mathcal{M}_0 \longrightarrow \mathcal{M}_0$ is defined by $E_X(\tilde{x}) = Tr_X(\gamma_X^* \, \tilde{x} \, \gamma_X)$, $\gamma_X \in \mathcal{M}_0$. E_X is the generalised conditional expectation of [9] . We give the proof of properties **(i) – (iv)** for the quantum dynamical semigroups.

Proof:

(i) Using the Taylor expansion of the exponential we write the dynamics as follows. $P_t^X(1) = e^{t\mathcal{L}^X}(1)$

$$P_t^X(1) = 1 + t\mathcal{L}^X(1) + \frac{t^2\mathcal{L}^X(\mathcal{L}^X(1))}{2} + \cdots$$

since $\mathcal{L}^{X,\Lambda}(1) = 0$, we have all the remaining terms to be zero hence , $P_t^X(1) = 1$

(ii) Now $\mathcal{L}^X\left(\int_0^t ds P_s^X\right) = P_t^X - P_0^X$

$$\begin{aligned}\langle (P_t^X - P_0^X)\tilde{x}, \tilde{y}\rangle &= \langle \mathcal{L}^X\left(\int_0^t ds P_s^X\right)\tilde{x}, \tilde{y}\rangle \\ &= \langle \tilde{x}, \mathcal{L}^X\left(\int_0^t ds P_s^X\right)^* \tilde{y}\rangle \\ &= \langle \tilde{x}, (P_t^X - P_0^X)\tilde{y}\rangle = \langle P_t^X(\tilde{x}), \tilde{y}\rangle = \langle \tilde{x}, P_t^X(\tilde{y})\rangle\end{aligned}$$

(iii) For contractivity of the semi-group, we have from [10] the following definition of the tangential functional. If $q \in (1, \infty)$, then for any $\tilde{x} \in L_q^+(\mathcal{M}_0)$, there exists a unique $\phi_p(\tilde{x}) \in L_p^+(\mathcal{M}_0)$ with $\frac{1}{p} + \frac{1}{q} = 1$ defined by

$$\phi_p(\tilde{x}) = \frac{\tilde{h}^{-\frac{1}{2p}}\left(\tilde{h}^{\frac{1}{2q}}\tilde{x}\tilde{h}^{\frac{1}{2q}}\right)^{\frac{q}{p}}\tilde{h}^{-\frac{1}{2p}}}{\|\tilde{x}\|_q^{q-2}} ,$$

and $\phi_p(0) = 0$ for $\tilde{x} = 0$, with the following properties;

(i) $\|\tilde{x}\|_{L_q(\mathcal{M}_0)}^2 = \langle \phi_p(\tilde{x}), \tilde{x}\rangle$, $\tilde{x} \in L_q^+(\mathcal{M}_0)$, and $\langle . , . \rangle$ is the duality pairing.

(ii) $\|\phi_p(\tilde{x})\|_{L_p(\mathcal{M}_0)} = \|\tilde{x}\|_{L_q(\mathcal{M}_0)}$ (iii) $\phi_p(|\tilde{x}|_q) = |\phi_p(\tilde{x})|_p$.

Now let $\tilde{x}, \tilde{y} \in L_p(\mathcal{M}_0)$, such that $\tilde{y} = P_t^X(\tilde{x})$, from (i), we have

$$\|\tilde{y}\|^2_{L_p(\mathcal{M}_0)} = \langle \phi_q(\tilde{x}), \tilde{y} \rangle \leq \left\|\phi_q(\tilde{x})\right\|_{L_q(\mathcal{M}_0)} \|\tilde{y}\|_{L_p(\mathcal{M}_0)}$$

$\|\tilde{y}\|_{L_p(\mathcal{M}_0)} \leq \left\|\phi_q(\tilde{x})\right\|_{L_q(\mathcal{M}_0)}$, replacing $\tilde{y}$ with $P_t^X(\tilde{x})$ and using property (ii) we have,

$$\|P_t^X(\tilde{x})\|_{L_p(\mathcal{M}_0)} \leq \left\|\phi_q(\tilde{x})\right\|_{L_q(\mathcal{M}_0)} = \|\tilde{x}\|_{L_p(\mathcal{M}_0)}$$

$$\|P_t^X(\tilde{x})\|_{L_p(\mathcal{M}_0)} \leq \|\tilde{x}\|_{L_p(\mathcal{M}_0)}.$$

(iv) $\varphi\left(P_t^X(\tilde{x})\right) = \langle 1, P_t^X(\tilde{x})\rangle = \langle P_t^X(1), (\tilde{x})\rangle = \langle 1, \tilde{x}\rangle = \varphi\ (\tilde{x})$

Q.E.D.

We define an infinite volume generator $\mathcal{L}^X$ formally by $\mathcal{L}^X = \sum_{j\in\mathbb{Z}^d} \mathcal{L}_{X+j}$ such that $\|\mathcal{L}^X\| < \infty$, where $\mathcal{L}_{X+j}(\tilde{x}) = E_{X+j}(\tilde{x}) - \tilde{x}$ is the elementary generator and E_{X+j} is a 2-positive unit preserving map such that $E_{X+j}\left(\widetilde{\mathcal{M}}_\Lambda\right) \subseteq \widetilde{\mathcal{M}}_{\Lambda^c+j}$. We start with the following definition of a normalized partial trace on $\widetilde{\mathcal{M}}_\Lambda$ that is, a completely positive map on $\widetilde{\mathcal{M}}_\Lambda$. We define this normalized partial trace Tr_j at a point $j \in \mathbb{Z}^d$ on the von Neumann algebra $\widetilde{\mathcal{M}}_\Lambda$. Since the point algebra $\widetilde{\mathcal{M}}_{\{j\}}$ generates $\widetilde{\mathcal{M}}_\Lambda$, we have

$$\widetilde{\mathcal{M}}_\Lambda = \widetilde{\mathcal{M}}_{\{j\}} \otimes \widetilde{\mathcal{M}}_{\{j\}^c}.$$

Definition 3

A normalized partial trace Tr_j on $\widetilde{\mathcal{M}}_\Lambda$ is a completely positive map $Tr_j : \widetilde{\mathcal{M}}_\Lambda \to \widetilde{\mathcal{M}}_{\{j\}^c}$ satisfying the following conditions for $\tilde{x} \in \mathcal{M}_\Lambda$,

(i) $Tr_j(\tilde{x}^* \tilde{x}) \geq 0$

(ii) $Tr_j(1) = 1$

(iii) $Tr_j\left(Tr_j\, \tilde{x}\right) = Tr_j(\tilde{x})$

(iv) $Tr_j(\tilde{x}\tilde{y}) = Tr_j(\tilde{y}\tilde{x}),\ \ \tilde{x}, \tilde{y} \in \mathcal{M}_\Lambda$

(v) $Tr_j\left(\tilde{g}.\tilde{x}.\tilde{f}\right) = \left(\tilde{g}.Tr_j(\tilde{x}).\tilde{f}\right),\ \ \tilde{x} \in \mathcal{M}_\Lambda$ and $\tilde{g}, \tilde{f} \in \widetilde{\mathcal{M}}_{\{j\}^c}$.

Definition 4

The discrete gradient $\partial_j \tilde{x}$ is defined by $\partial_j \tilde{x} = \tilde{x} - Tr_j \tilde{x}$, for a vector $\boldsymbol{j} \in \mathbb{Z}^d$.

This defines a seminorm $|||.|||$ on $\mathcal{M}_0$ given by $|||\tilde{x}||| \equiv \sum_{j\in\mathbb{Z}^d}\left\|\partial_j \tilde{x}\right\|$.

Let the set of operators in $\mathcal{M}_0$ with finite seminorm $|||.|||$ be denoted by $\mathcal{M}_1$, that is, the set $\mathcal{M}_1 = \{\tilde{x}:\ \tilde{x} \in \mathcal{M}_0\, , |||\tilde{x}||| < \infty\ \}$.

Definition 5

For any $\tilde{x} \in \mathcal{M}_1$ an elementary operator $\mathcal{L}_{X+j}$ is called **regular** if there is a positive constant b_{jk} with $\boldsymbol{j}, \boldsymbol{k} \in \mathbb{Z}^d$ such that $\left\|\mathcal{L}_{X+j}\tilde{x}\right\| \leq \sum_k b_{jk}\left\|\partial_j \tilde{x}\right\|$ and $b_{jk} \in [0, \infty)$, such that $sup_j \sum_k b_{jk} < \infty$.

Remark: The stochastic dynamics sequence P_t^{X,Λ_n} is Cauchy in the norm topology for the sets of increasing bounded regions Λ_n satisfying $\Lambda_{n+1} \supset \Lambda_n$ and $\cup\Lambda_n \equiv \mathbb{Z}^d$. The limit exists as $\Lambda_n \to \infty$ and defines an extended volume quantum stochastic dynamics P_t^X on $\mathcal{M}_0$, provided the elementary generator $\mathcal{L}_{X+j},\ j \in \mathbb{Z}^d$ is regular and satisfies some certain conditions . We now give a proof of the main theorem, that is, the P_t^X extended dynamical semigroup is strongly ergodic

Theorem :

The semi-group $(P_t^X)_{t\geq 0}$ is strongly ergodic in the sense that there is a unique $(P_t^X)_{t\geq 0}$ - invariant locally normal state ω for which we have,

$$|||P_t^X(\tilde{x}) - \omega(\tilde{x})||| \leq 2e^{-(1-\lambda)|X|t}|||\tilde{x}||| \ ,\ \ \tilde{x} \in \mathcal{M}_1 .$$

Proof

To show the strong ergodicity property of the dynamics P_t^X, we have the following formulation. We note that by the weak compactness of the space of state on $\mathcal{M}_\Lambda$ and the fact that the dynamics P_t^X has a Feller property, the set of invariant states with respect to the dynamics is non-empty. Let ω be such an invariant locally normal state, then $\quad |||P_t^X(\tilde{x}) - \omega(\tilde{x})||| = \|P_t^X(\tilde{x}) - \omega(P_t^X(\tilde{x}))\|$

now we consider the tensor product algebra of $\mathcal{M}_\Lambda$ by itself, and, from [4], the completely positive map

$\theta : \mathcal{M}_0 \otimes \mathcal{M}_0 \longrightarrow \mathcal{M}_0$ defined by $\theta\left(\tilde{x}_{\Lambda_1} \otimes \tilde{x}_{\Lambda_2}\right) = \varphi_\Lambda(\tilde{x}_{\Lambda_1})\tilde{x}_{\Lambda_2}$. $\tilde{x}_{\Lambda_1}, \tilde{x}_{\Lambda_2} \in \mathcal{M}_0$.

We note that $\theta(P_t^X(\tilde{x}) \otimes I) = \omega(P_t^X(\tilde{x}))I = \omega(P_t^X(\tilde{x}))$

$\theta(I \otimes P_t^X(\tilde{x})) = \omega(I)P_t^X(\tilde{x}) = P_t^X(\tilde{x})$,

since $\omega \circ P_t^X = \omega$ we have,

$$\begin{aligned} \|P_t^X(\tilde{x}) - \omega(\tilde{x})\| &= \|P_t^X(\tilde{x}) - \omega(P_t^X(\tilde{x}))\| \\ &= \|\theta(I \otimes P_t^X(\tilde{x})) - \theta(P_t^X(\tilde{x}) \otimes I)\| \\ &\le \|\theta(I \otimes P_t^X(\tilde{x}) - P_t^X(\tilde{x}) \otimes I)\| \\ &\le \|I \otimes P_t^X(\tilde{x}) - P_t^X(\tilde{x}) \otimes I\|. \end{aligned}$$

With this formulation in mind, we may express $P_t^X(\tilde{x})$ as follows:

$$\begin{aligned} P_t^X(\tilde{x}) = P_t^X(\tilde{x}) &+ Tr_{j_1}P_t^X(\tilde{x}) - Tr_{j_1}P_t^X(\tilde{x}) + Tr_{j_2}P_t^X(\tilde{x}) - Tr_{j_2}P_t^X(\tilde{x}) + Tr_{j_3}P_t^X(\tilde{x}) \\ &- Tr_{j_3}P_t^X(\tilde{x}) + Tr_{j_4}P_t^X(\tilde{x}) - Tr_{j_4}P_t^X(\tilde{x}) + \ldots + Tr_{j_n}P_t^X(\tilde{x}) - Tr_{j_n}P_t^X(\tilde{x}) \end{aligned}$$

thus let $\{j_n\}_{n\in\mathbb{N}} \subset \mathbb{Z}^d$ be a sequence with lexicographic ordering such that for each $j_i \in \Lambda_i$ and $\Lambda_{i-1} \subset \Lambda_i$ we have $j_{i-1} \le j_i$. Since the partial traces $Tr_{j_{i-1}}$ and Tr_{j_i} are projections with the ordering $Tr_{j_1} \le Tr_{j_2}$ we have the relation

$$Tr_{j_i}Tr_{j_{i-1}} = Tr_{j_{i-1}}Tr_{j_i} = Tr_{j_{i-1}},$$

hence we rewrite the zero terms as follows

$$\begin{aligned} P_t^X(\tilde{x}) = P_t^X(\tilde{x}) &+ Tr_{j_1}P_t^X(\tilde{x}) - Tr_{j_1}Tr_{j_2}P_t^X(\tilde{x}) + Tr_{j_2}P_t^X(\tilde{x}) - Tr_{j_2}Tr_{j_3}P_t^X(\tilde{x}) \\ &+ Tr_{j_3}P_t^X(\tilde{x}) - Tr_{j_3}Tr_{j_4}P_t^X(\tilde{x}) \;\ldots\; + Tr_{j_n}P_t^X(\tilde{x}) - Tr_{j_n}Tr_{j_{n+1}}P_t^X(\tilde{x}). \end{aligned}$$

$$\begin{aligned} P_t^X(\tilde{x}) = P_t^X(\tilde{x}) &- Tr_{j_1}P_t^X(\tilde{x}) + Tr_{j_1}P_t^X(\tilde{x}) + Tr_{j_1}(P_t^X(\tilde{x}) - Tr_{j_2}P_t^X(\tilde{x})) \\ &+ Tr_{j_2}\left(P_t^X(\tilde{x}) - Tr_{j_3}P_t^X(\tilde{x})\right) + Tr_{j_3}\left(P_t^X(\tilde{x}) - Tr_{j_4}P_t^X(\tilde{x})\right) \\ &+ Tr_{j_4}(P_t^X(\tilde{x}) - Tr_{j_5}P_t^X(\tilde{x})) \ldots \; + Tr_{j_n}(P_t^X(\tilde{x}) - Tr_{j_{n+1}}P_t^X(\tilde{x})) \end{aligned}$$

hence from $\quad \partial_{j_1}P_t^X(\tilde{x}) = P_t^X(\tilde{x}) - Tr_{j_1}P_t^X(\tilde{x})$,

we have $\quad P_t^X(\tilde{x}) = \partial_{j_1}P_t^X(\tilde{x}) + \sum_{n\in\mathbb{N}} Tr_{\{j_1,j_2,j_3,\ldots,j_n\}}\left(P_t^X(\tilde{x}) - Tr_{j_{n+1}}P_t^X(\tilde{x})\right)$

note that the summation is finite because ,

$$\begin{aligned} Tr_{j_n}(P_t^X(\tilde{x}) - Tr_{j_{n+1}}P_t^X(\tilde{x})) &= Tr_{j_n}P_t^X(\tilde{x}) - Tr_{j_n}Tr_{j_{n+1}}P_t^X(\tilde{x}) \\ &= Tr_{j_n}P_t^X(\tilde{x}) - Tr_{j_n}P_t^X(\tilde{x}) = 0 \quad \text{for } n \in \mathbb{N}. \end{aligned}$$

Thus we have

$$\sum_{n\in\mathbb{N}} \left\| Tr_{\{j_1,j_2,j_3,\ldots,j_n\}}\left(P_t^X(\tilde{x}) - Tr_{j_{n+1}}P_t^X(\tilde{x})\right) \right\| = \sum_{n\in\mathbb{N}} \left\| Tr_{\{j_1\ldots j_n\}}\partial_{j_{n+1}}P_t^X(\tilde{x}) \right\| < \infty$$

therefore we can write $\quad I \otimes P_t^X(\tilde{x}) = \partial_{j_1}P_t^X(\tilde{x}) + \sum_{n\in\mathbb{N}} Tr_{\{j_1\ldots j_n\}}\partial_{j_{n+1}}P_t^X(\tilde{x})$

and also we have $\quad P_t^X(\tilde{x}) \otimes I = \sum_{n\in\mathbb{N}} Tr_{\{j_1\ldots j_n\}}\partial_{j_{n+1}}P_t^X(\tilde{x}) - \partial_{j_1}P_t^X(\tilde{x})$

hence, $\quad \|P_t^X(\tilde{x}) - \omega(\tilde{x})\| \le \left\|\left(I \otimes P_t^X(\tilde{x})\right) - (P_t^X(\tilde{x}) \otimes I)\right\|$

$$\le \left\| \left(\partial_{j_1}P_t^X(\tilde{x}) + \sum_{n\in\mathbb{N}} Tr_{\{j_1\ldots j_n\}}\partial_{j_{n+1}}P_t^X(\tilde{x})\right) - \left(\sum_{n\in\mathbb{N}} Tr_{\{j_1\ldots j_n\}}\partial_{j_{n+1}}P_t^X(\tilde{x}) - \partial_{j_1}P_t^X(\tilde{x})\right) \right\|$$

$$\le 2\left\|\partial_{j1}P_t^x(\tilde{x})\right\| = 2\left|\left\|P_t^X(\tilde{x})\right\|\right| \le 2\, e^{-(1-\lambda)|X|t}|||\tilde{x}|||$$

4.0 Concluding Remark:

The extended quantum dynamics defined as the thermodynamic limit of the finite volume evolution $\square^{\square,\Lambda\square}_{\square}$ is shown to be ergodic, if it exists in the appropriate topology. This is quite important in any serious study of stochastic processes and expectation semigroups, especially the algebraic approach to quantum statistical mechanics.

Dedication:
The author would like to dedicate this work to the immediate past president of the Nigerian Association of Mathematical Physics, the late Prof R O Ayeni.

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