Journal of the Nigerian Association of Mathematical Physics
Volume 23 (March, 2013), pp 61 – 64
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Dihedral Group (Dn): A Case Study of a Finite Group(An in-depth Approach)

Ademola, Lois Adewoye.

Department Of mathematics,
University Of Jos, Jos, Nigeria.

Abstract

Today the Dihedral groups turn out to be one of the simplest examples of finite groups. The role the size of the conjugacy class plays on the finite groups is a major and striving area of research[1]. This paper will consider the dihedral groups via its generators and relations, where a will be considered as a generator of order n and b a generator of order 2. These two generators are taken to satisfy the relation $ba=a^{-1}b$. Particular interests is given to the order, cosets, normalizers, and conjugacy classes of this group. The conclusion is drawn from the intelligent and detailed report derived from the results obtained.

1.0 Introduction

The Dihedral groups are groups of symmetries gotten by considering the rotations and reflections of regular polygons. In general today it is well known that when considering groups that they can be shown to be dihedral groups by considering how these groups are generated. In particular, groups generated by functions whose inverse is itself, are dihedral groups. We shall in this paper take a preview of dihedral groups by considering some particular dihedral groups and how their structures are influenced by their Cosets, Abelian, Normal, Conjugacy and Centralizer properties.

2.0 Basic Definitions and Preliminary Results

Definition 1:

A non-empty set G with one algebraic operation $\bullet$, is called a *group* if the following conditions are satisfied:

1. CLOSURE: If $\mathbf{a}$ and $\mathbf{b}$ are in G, then $\mathbf{a} \bullet \mathbf{b}$ is also in G.

2. ASSOCIATIVITY: If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are in G then $(\mathbf{a} \bullet \mathbf{b}) \bullet \mathbf{c} = \mathbf{a} \bullet (\mathbf{b} \bullet \mathbf{c})$.

3. IDENTITY: There is an element $\mathbf{e}$ of G such that for any element $\mathbf{a}$ of G

$\mathbf{a} \bullet \mathbf{e} = \mathbf{e} \bullet \mathbf{a} = \mathbf{a}$.

4. INVERSES: For any element **a** of $\mathbf{G}$ there is an element $\mathbf{a}^{-1}$ such that $\mathbf{a} \bullet \mathbf{a}^{-1} = \mathbf{e}$ and $\mathbf{a}^{-1} \bullet \mathbf{a} = \mathbf{e}$

The operation in G need not be commutative. If it is commutative, then G is called a commutative or abelian group [2] .

Definition 2

Given a group G, a subgroup $K \leq G$ is a *normal* subgroup, denoted $K \lhd G$, if

$gKg^{-1} = K$ for every $g \in G$ [3] .

Definition 3

Given a group G and an element g of G. The least positive integer m such that $g^m = 1$ is called the *order* of g. Now if all the elements of G are exhausted by g, we say the group G is *generated* by g. Denoted $G = \langle g \rangle$[4].

Definition 4

The *Dihedral groups* can be defined as:

$D_n = \langle a, b | a^n = 1, b^2 = 1, ba = a^{-1}b \rangle$

Definition 5

Given a group G, H a subgroup of of G and $g \in G$. Then

$$gH = \{gh | h \in H\}$$

is called a left *Coset* of H in G [4].

Corresponding author: ***Lois Adewoye,*** E-mail: loisola@yahoo.com-, Tel. +234 8072162670

Definition 6

Let G be a group, for $x, y \in G$, we say that y is a conjugate of x, if there exist $g \in G$, such that $y = gxg^{-1}$. written $x \sim y$. The equivalence classes of $\sim$ are called te conjugacy classes of G [5].

3. Proof of Main Result

I. Identities and cosets of the D_n

Using the definition of the dihedral groups as given in definition 2 above, we shall here verify the identities that:

a. $ba^r = a^{-r}b \text{ for all } 1 \leq r \leq n$

b. Every element of the form $a^r b$ has an order of 2.

Proof of a:

We shall prove this inductively.

Firstly we assume that $ba^k = a^{-k}b$, so that we have:

$ba^{k+1} = ba^k a$

$= a^{-k}ba$

$= a^{-k}a^{-1}b$

$= a^{-(k+1)}b$

Thus we conclude that this identity holds for all $r \in Z^+$, where Z^+ is the set of all positive integers.

Proof of b:

To show that every element with the form $a^r b$ has an order of 2, we simply consider the the form as below:

$(a^r b)^2 = a^r ba^r b$

But from Proof of a we showed that $ba^r = a^{-r}b$, thus

$(a^r b)^2 = a^r a^{-r}bb$

$= a^{r+(-r)}b^2$

$= a^0 b^2$

Also in definition 4, $b^2 = 1$ and it is obvious that $a^0 = 1$. Therefore we get that:

$(a^r b)^2 = 1$

Next we shall consider how to find all the left cosets of b. Firstly we know from the definition of a D_n that $b^2 = 1$

We also, in particular know from definition 3, that b is a generator. Thus we seek to find all the left cosets of $\langle b \rangle$. Now in D_n the only other element is a, which has an order n. Thus we expect the left cosets of $\langle b \rangle$ to take te form:

$a^r\langle b \rangle = \{a^r, a^r b\}, r \in Z^+$

Similarly we would have all its right coset to take the form;

$\langle b \rangle a^r = \{a^r, a^{-r}b\}, r \in Z^+$

II.The Normal and Abelian properties of D_n.

We shall consider two specific Dihedral groups with respect to their being normal and also being abelian.

Case1: Given the group D_6 and a subgroup $\langle a^3 \rangle = \{e, a^3\}$ of D_6.

We will here verify that

a. $\langle a^3 \rangle$ is a normal subgroup of D_6

b. $D_6/\langle a^3 \rangle$ is abelian. Where $D_6/\langle a^3 \rangle$:= the set of all left cosets of $\langle a^3 \rangle$ in D_6.

Proof of a:

To prove a we shall consider two methods.

Method1:

Using definition 2, $gkg^{-1} \in \langle a^3 \rangle$ where k is now an element of $\langle a^3 \rangle$ and $g \in D_6$

From the identities shown above, we can deduce that

$ba^3 = a^{-3}b = a^3 b$

So

$bkb^{-1} = k$

If $k = a^3$.

The next but only element of $\langle a^3 \rangle$ to check is identity we we shall denote e, now a will always commute with a^3 irrespective of the value of n. Thus, $gkg^{-1} \in \langle a^3 \rangle$ with $k \in \langle a^3 \rangle$ and $g \in D_6$.

Method 2:

We shall in this method get all the left and right cosets of $K = \langle a^3 \rangle$ and compare.

The left cosets are as follows:

$K = \{e, a^3\}$
$aK = a\{e, a^3\} = \{a, a^4\}$
$a^2K = a^2\{e, a^3\} = \{a^2, a^5\}$
$bK = b\{e, a^3\} = \{b, ba^3\}$
$abK = ab\{e, a^3\} = \{ab, aba^3\} = \{a, aa^3 b\} = \{a, a^4 b\}$
$a^2 bK = a^2 b\{e, a^3\} = \{a^2 b, a^2 ba^3\} = \{a^2 b, a^2 a^3 b\} = \{a^2 b, a^5 b\}$

Note that we were able to get the results above using the identity $ba^r = a^{-r} b \; for \; all \; 1 \leq r \leq n$

The right cosets are as follows:

$K = \{e, a^3\}$
$Ka = \{e, a^3\}a = \{a, a^4\}$
$Ka^2 = \{e, a^3\}a^2 = \{a^2, a^5\}$
$Kb = \{e, a^3\}b = \{b, a^3 b\}$
$Kab = \{e, a^3\}ab = \{ab, a^3 ab\} = \{a, a^4 b\}$
$Ka^2 b = \{e, a^3\}a^2 b = \{a^2 b, a^3 a^2 b\} = \{a^2 b, a^5 b\}$

Thus we have verified the fact that $\langle a^3 \rangle$ is a normal subgroup of D_6.

Proof of b:

To show that $D_6/\langle a^3 \rangle$ is abelian, we will need to show that $(aK)(bK) = (bK)(aK)$.

Now from the results gotten for the proof of a above, we get that:

$aK = \{a, a^4\}$
$bK = \{b, ba^3\}$

Thus,

$(aK)(bK) = abK = \{ab, a^4 b\}$

And

$(bK)(aK) = baK = a^5 bK = \{a^5 b, a^2 b\}$

True because by definition 4, $a^6 = 1$.

Therefore $(aK)(bK) \neq (bK)(aK)$. Thus we conclude that $D_6/\langle a^3 \rangle$ is not abelian.

Case 2:

Given the group D_{12} and a subgroup $K = \{e, a^3, a^6, a^9\}$ of D_{12}.

We will here verify that

a. K is a normal subgroup of D_{12}
b. D_{12}/K is abelian.

Proof of a:

In case1, we showed that $\langle a^3 \rangle$ is a normal subgroup. In fact it is obviously a subgroup of K, we can show here that K is normal by using the identity $ba^r = a^{-r} b$.

We see that :

$a^r (a^{3m}) a^{-r} = a^{3m}$

Also,

$a^r b(a^{3m}) a^{-r} b = a^r a^{-3m} a^{-r} = (a^{-3m})^{-1}$

Next we consider the cosets which include the following:

$K = \{e, a^3, a^6, a^9\}$
$Kb = \{b, a^3 b, a^6 b, a^9 b\}$
$Ka = \{a, a^4, a^7, a^{10}\}$
$Kab = \{ab, a^4 b, a^7 b, a^{10} b\}$
$Ka^2 = \{a^2, a^5, a^8, a^{11}\}$
$Ka^2 b = \{a^2 b, a^5 b, a^8 b, a^{11} b\}$

Proof of b:

In proving this we again consider $(Ka)(Kb) = (Kb)(Ka)$.

So

$(Ka)(Kb) = Kab$

But

$(Kb)(Ka) = Ka^2b$

And $ba = a^{11}b$ in Ka^2b.

III.The Conjugacy and Centralizer properties of D_n.

Just as in 2, we shall consider two specific Dihedral groups with respect to their conjugacy classes and their centralizers.

Case1:

We shall here obtain all the cojugacy classes of D_4.

Using definition 4, let

$D_4 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$

Such that $a^4 = e, b^2 = e$ and $ba = a^{-1}b$.

Now since $yey^{-1} = e$, it means that e is a conjugate of itself.

Now if m is any chosen power of a, then y would commute with a, so that $yay^{-1} = a$

Furthermore if $y = a^ib$, then we would have that

$yay^{-1} = a^ibaa^{-i}b = a^ia^{i-1}b^2 = a^{2i-1}$

Thus in D_4, a^3 and a are the only conjugates for a.

Now from 1 and 2 above, we are able to deduce that in D_4

$ya^2y^{-1} = a^2$

Thus, a^2 is conjugated by no other element.

Next, if $y = a^i$,then we expect that

$yby^{-1} = a^iba^{-i} = a^ia^ib \quad = a^{2i}b$

If $y = a^ib$,then we expect that

$yby^{-1} = a^ibba^{-i}b = a^ia^ib \quad = a^{2i}b$

So the only conjugate of b is a^2b.

$yby^{-1} = a^iba^{-i} = a^ia^ib \quad = a^{2i}b$

If $y = a^ib$,then we expect that

$yaby^{-1} = a^ibab\,a^{-i}b = a^{i+1}a^ib \quad = a^{2i+1}b$

So the only conjugate of ab is a^3b.

Case2:

We shall here find the centralizer of a in D_n.

The centralizer $C(a)$ by definition would contain every power of a, so that $\langle a\rangle \subseteq C(a)$. This means that there are atleast n elements in $C(a)$.

Conversely $C(a) \neq D_n$, because by the definition of $C(a)$, $b \in C(a)$. Now half the elements of D_n are in $\langle a\rangle$ and by lagrange's theorem there are no subgroups lying strictly between D_n and $\langle a\rangle$, hence we have that

$\langle a\rangle \subseteq C(a)$

And

$C(a) \subseteq D_n$

but $C(a) \neq D_n$, so we get that

$C(a) = \langle a\rangle$.

Conclusion

With the above results derived from consideration of specific cases of the Dihedral group, an intelligent report has been generated which will birth further observations in subsequent papers.

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