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Homotopy Analysis Method Solution of SIR Model

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Abstract

The Homotopy Analysis Method (HAM) is an analytic method developed to solve non-linear problems. We have expatiated on the theory behind the method and applied to it a closed group SIR problem to determine the magnitude of an epidemic for a hypothetic case. We considered cases when the basic reproduction number R_0 is less than 1 (when there is no epidemic) and when R_0 is greater than 1 (an epidemic case) and the results are presented graphically.

Keywords: solar, battery, charging, GSM, generator,

1.0 Introduction

The Homotopy Analysis Method (HAM) is an analytic method developed to solve non-linear problems. It was first proposed by Liao in 1992 [1] using the concept of Topology. The HAM has advantage over the numerical methods because it produces series solutions which are continuous unlike the numerical methods that approximate solutions at discrete points, giving rise to rounding off errors. The HAM has been applied to solve many types of nonlinear problems: integral and integro-differential equations [2], system of ODEs [3], coupled equations [4,5] and many others [6 -11]. The method is independent of any small and large physical parameter. It provides us with a simple way to ensure the convergence of the solution by using the convergence control parameter to adjust and control the convergence region and the rate of convergence of approximate series. We established the reliability and efficiency of the method by applying it to the problem of Susceptible, Infectious and Removed, the SIR model of a disease

2.0 Basic Idea of the HAM

To illustrate the basic idea of the HAM, consider the following differential equation

$$N(U(x,t)) = 0 \qquad\qquad U_0 = \alpha \qquad ... \qquad (1)$$

where N is a nonlinear operator, x, t the independent variables of the exact solution $U(x,t)$ to be determined. $U(x,t)$ can be a vector. Where α is a constant. Let $U_0(x,t)$ denote an initial approximation or initial guess of $U_0(x,t)$, which must satisfy the initial or boundary condition of the given problem (1), $h \neq 0$ be nonzero auxiliary parameter, $Y(t) \neq 0$ an auxiliary function and L an auxiliary linear operator. Liao constructed the so called zeroth-order deformation equation given as

$$(1-q)L(\Phi(x,t,q) - U_0(x,t)) = qhH(x,t)N(\Phi(x,t,q)) \qquad ... \qquad (2)$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero parameter called the convergence control parameter, $H(x,t) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $U_0(x,t)$ is an initial guess of $U(x,t)$, which must satisfy the initial conditions from the given problem, $\Phi(x,t,q)$ is an unknown function which is defined in terms of $U(x,t)$. It is important to know that one has great freedom to choose auxiliary linear operator L, the

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auxiliary parameter h and the auxiliary function Y(t)

Obviously, when $q=0$ and $p=1$ in (2), we have

$$\Phi(x,\ t,\ 0)=U_0(x,\ t) \qquad \ldots \qquad (3)$$

and

$$\Phi(x,\ t,\ 1)=U(x,\ t) \qquad \ldots \qquad (4)$$

Thus as p increases from 0 to 1, $\Phi(x,\ t,\ q)$ deforms (varies) continuously from the initial approximation $U_o(x,\ t)$ to the exact solution $U(x,\ t)$

This kind of continuous variation is called deformation in topology. Expanding $\Phi(x,\ t,\ q)$ in Taylor's series with respect to the embedding parameter q, one has

$$\Phi(x,\ t,\ q)=U_0(x,\ t)+\sum_{k=1}^{\infty}U_k(x,\ t)q^k \qquad \ldots \qquad (5)$$

Where $U_k(x,\ t)=\frac{1}{k}\frac{\partial^k}{\partial q^k}\Phi(x,\ t,\ q)\Big|_{q=0}$ ... (6)

If the auxiliary linear operator L, the initial guess $U_o(t)$, the convergence control parameter h and the auxiliary function $Y(t)$ are so properly chosen that equation (6) converges at $q=1$ we have

$$U(x,\ t)=U_0(x,\ t)+\sum_{k=1}^{\infty}U_k(x,\ t) \qquad \ldots \qquad (7)$$

which must be one of the solutions of the original non-linear equation as proved by Liao [12,13]

We defined the vector

$$\vec{U}_{k-1}=U_0(x,\ t),\ U_1(x,\ t),\ U_2(x,\ t),\ \ldots\ U_{k-1}(x,\ t)$$

According to the definition of equation (5), the equation of $U_k(x,\ t)$ can be defined from the zeroth order deformation of equation (2) by differentiating it k times with respect to q and then dividing by k! And finally setting q = 0, we have the so called kth order deformation equation

$$L(U_k(x,\ t))-X_kU_{k-1}(x,\ t)=hH(x,\ t)R_k\left(\vec{U}_{k-1}(x,\ t)\right) \qquad \ldots \qquad (8)$$

Where

$$R_k\left(\vec{U}_{k-1}(x,\ t)\right)=\frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial q^{k-1}}N(\Phi(x,\ t,\ q))\Big|_{q=0} \qquad \ldots \qquad (9)$$

And

$$X_k=\begin{cases}0 & k\le 1\\ 1 & k>1\end{cases}$$

Applying the inverse linear operator L^{-1} to both sides of equation (8), we have

$$U_k(x,\ t)=X_kU_{k-1}(x,\ t)+hL^{-1}H(x,\ t)R_k\left(\vec{U}_{k-1}(x,\ t)\right) \qquad \ldots \qquad (10)$$

Note that the higher order deformation equation (8) is governed by the linear operator L and the term $R_k\left(\vec{U}_{k-1}(x,\ t)\right)$ is expressed by equation (9) for any non-linear operator.

From equation (10) it is easy to obtain $U_k(x,\ t)$ for $k\ge 1$. At the kth order, we have

$$U(x,\ t)=\sum_{k=0}^{\infty}U_k(x,\ t) \qquad \ldots \qquad (11)$$

Equation (11) will give accurate approximation of the original equation (1) if equation (1) admits a unique solution, but if not, then it will give solution among many other possible solutions.

Note that equation (11) can be expressed for any given nonlinear operator N and we can use Mathematica and Maple to get the mth-order derivative U_m(t).

3.0 Application of HAM to SIR Model

An SIR model is an epidemiological model that computes the theoretical number of people infected with a contagious disease in a close population over time. The model was created in 1927 by Kermack and Mckendrick [14] on which they considered a closed population with only three compartments; susceptible S(r), infected I(r), and recovered R(r).

The model can be represented diagrammatically by

$$S \rightarrow I \rightarrow R$$

Using a closed population N that is neglecting birth and disease-related death, we have

$$N = S(r) + I(r) + R(r) \qquad \ldots \qquad (12)$$

Where N is the number of population at time r

Kermack and Mckendrick [14] derived the following equations

$$\frac{ds}{dr} = -\beta IS \qquad \ldots \qquad (13)$$

$$\frac{dI}{dr} = \beta IS - \gamma I \qquad \ldots \qquad (14)$$

$$\frac{dR}{dr} = \gamma I \qquad \ldots \qquad (15)$$

Where γ is the rate of recovery and β is the infectious contact rate.

Using non-dimensional variables define as

$$u = \frac{S}{N}, \quad v = \frac{I}{N}, \quad w = \frac{R}{N}, \quad t = \gamma r$$

We choose our initial conditions in such a way that

$$u(0) + v(0) + w(0) = 1$$

and assume our initial conditions to be

$$u(0) = 0.9, \; v(0) = 0.1, \quad w(0) = 0$$

We have the non-dimensional equations of SIR model given by

$$\frac{du}{dt} = -R_o uv \qquad \ldots \qquad (16)$$

$$\frac{dv}{dt} = R_o uv - v \qquad \ldots \qquad (17)$$

$$\frac{dw}{dt} = v \qquad \ldots \qquad (18)$$

where the basic reproduction number R_0 is defined as $R_0 = \frac{\beta}{\gamma}$

Applying HAM to those equations (16), (17) and (18), we defined the nonlinear operators

$$N[\phi_1(t,p)] = \frac{\partial \phi_1(t,p)}{\partial t} + R_o \phi_1(t,p)\phi_2(t,p) \qquad \ldots \qquad (19)$$

$$N[\phi_2(t,p)] = \frac{\partial \phi_2(t,p)}{\partial t} - R_o \phi_1(t,p)\phi_2(t,p) + \phi_2(t,p) \qquad \ldots \qquad (20)$$

$$N[\phi_3(t,p)] = \frac{\partial \phi_3(t,p)}{\partial t} - \phi_2(t,p) \qquad \ldots \qquad (21)$$

and initial approximations

$$u_o(t) = 0.9 \qquad v_o(t) = 0.1e^{-t}, \quad w_o(t) = 0$$

The auxiliary linear operators are given by

$$L_1(\phi_1(t,p)) = \frac{\partial \phi_1(t,p)}{\partial t} \quad \ldots \quad (22)$$

$$L_2[\phi_2(t,p)] = \frac{\partial \phi_2(t,p)}{\partial t} + \phi_2(t,p) \quad \ldots \quad (23)$$

$$L_2[\phi_2(t,p)] = \frac{\partial \phi_3(t,p)}{\partial t} \quad \ldots \quad (24)$$

With the properly that

$$L_1(C_1) = 0$$

$$L_2(C_2 e^{-t}) = 0$$

$$L_3(C_3) = 0$$

where $C_i, (i = 1,2,3)$ are integral constants to be determined.

Letting $p \in [0,1]$ denotes the embedding parameter and h_1, h_2, h_3 are non-zero auxiliary parameters and $Y_i(t)$ $(i = 1,2,3)$ are the nonzero auxiliary functions. We then construct the following zero-order deformation equations given by

$$(1-p)L_1[\phi_1(t,p) - u_o(t)] = h_1 p Y_1(t) N[\phi_1(t,p)] \quad \ldots \quad (25)$$

$$(1-p)L_2[\phi_2(t,p) - v_o(t)] = h_2 p Y_2(t) N[\phi_2(t,p)] \quad \ldots \quad (26)$$

$$(1-p)L_3[\phi_3(t,p) - w_o(t)] = h_3 p Y_3(t) N[\phi_3(t,p)] \quad \ldots \quad (27)$$

(25), (26) and (27) are subject to the initial conditions

$$\phi_1(0,p) = 0.9 \quad \ldots \quad (28)$$

$$\phi_2(0,p) = 0.1 \quad \ldots \quad (29)$$

$$\phi_3(0,p) = 0 \quad \ldots \quad (30)$$

Where $\phi_i(t,p)$ (i=1, 2, 3) are functions of t and p (embedding parameter)

From (25), (26) and (27), we have when p=0

$$\phi_1(t,0) = u_o(t) \quad \ldots \quad (28a)$$

$$\phi_2(t,0) = v_o(t) \quad \ldots \quad (29a)$$

$$\phi_3(t,0) = w_o(t) \quad \ldots \quad (30a)$$

And

p =1

$$\phi_1(t,1) = u(t) \quad \ldots \quad (28b)$$

$$\phi_2(t,1) = v(t) \quad \ldots \quad (29b)$$

$$\phi_3(t,1) = w(t) \quad \ldots \quad (30b)$$

So as p increases from 0 to 1, $\phi_i(t,p)$ $(i = 1,2,3)$ vary from $u_0(t)$, $v_0(t)$, $w_0(t)$ to $u_1(t)$, $v_1(t)$, $w_1(t)$.

Expanding $\phi_1(t,p), \phi_2(t,p)$ *and* $\phi_3(t,p)$ with respect to p by using the Taylor's theorem and imploring (28a), (29a) and (30a).We have

$$\phi_1(t,p) = u_o(t) + \sum_{m=1}^{+\infty} u_m(t) p^m \quad \ldots \quad (31)$$

$$\phi_2(t,p) = v_o(t) + \sum_{m=1}^{+\infty} v_m(t) p^m \quad \ldots \quad (32)$$

$$\phi_3(t,p)=w_o(t)+\sum_{m=1}^{+\infty}w_m(t)p^m \quad \ldots \quad (33)$$

Where

$$u_m(t)=\frac{1}{m!}\frac{\partial^m\phi_1(t,p)}{\partial p^m}\bigg|_{p=0}$$

$$v_m(t)=\frac{1}{m!}\frac{\partial^m\phi_2(t,p)}{\partial p^m}\bigg|_{p=0}$$

$$w_m(t)=\frac{1}{m!}\frac{\partial^m\phi_3(t,p)}{\partial p^m}\bigg|_{p=0}$$

If the nonzero auxiliary parameters h_1, h_2, h_3 and auxiliary functions $Y_1(t)$, $Y_2(t)$ and $Y_3(t)$ are properly chosen in such a way that these series (31)-(33) are convergent at p=1, we have from

$$u(t)=u_o(t)+\sum_{m=1}^{+\infty}u_m(t) \quad \ldots \quad (34)$$

$$v(t)=v_o(t)+\sum_{m=1}^{+\infty}v_m(t) \quad \ldots \quad (35)$$

$$w(t)=w_o(t)+\sum_{m=1}^{+\infty}w_m(t) \quad \ldots \quad (36)$$

Equation (34) – (36) are called the homotopy series solution for u(t), v(t) and w(t)

M^{th} –order deformation equations

Let us defined the vectors

$$\vec{u}_m=[u_1(t),u_2(t),\ldots,u_m(t)]$$
$$\vec{v}_m=[v_1(t),v_2(t),\ldots,v_m(t)] \qquad m\geq 1$$
$$\overline{w}_m=[w_1(t),w_2(t),\ldots,w_m(t)]$$

We have the m^{th}-order deformation equations given as

$$L_1[u_m(t)-\chi_m u_{m-1}(t)]=h_1Y_1(t)R_m(\vec{u}_{m-1}) \quad \ldots \quad (37)$$

$$u_m(0)=0 \quad \ldots \quad (37a)$$

$$L_2[v_m(t)-\chi_m v_{m-1}(t)]=h_2Y_2(t)R_m(\vec{v}_{m-1}) \quad \ldots \quad (38)$$

$$v_m(0)=0 \quad \ldots \quad (38a)$$

$$L_3[w_m(t)-\chi_m w_{m-1}(t)]=h_3Y_3(t)R_m(\vec{w}_{m-1}) \quad \ldots \quad (39)$$

$$w_m(0)=0 \quad \ldots \quad (39a)$$

Taking the inverse auxiliary linear operator of both sides of (37) - (39), we have

$$u_m(t)=\chi_m u_{m-1}(t)+h_1L_1^{-1}[Y_1(t)R_m(\vec{u}_{m-1})]+C_1 \quad \ldots \quad (40)$$

$$v_m(t)=\chi_m v_{m-1}(t)+h_2L_2^{-1}[Y_2(t)R_m(\vec{v}_{m-1})]+C_2e^{-t} \quad \ldots \quad (41)$$

$$w_m(t)=\chi_m w_{m-1}(t)+h_3L_3^{-1}[Y_3(t)R_m(\vec{w}_{m-1})]+C_3 \quad \ldots \quad (42)$$

Let h_1=h_2 =h_3,=h for simplicity. Then from the definition of our auxiliary linear operator of each function. We have the inverse auxiliary linear operators defined as

$$L_1^{-1} = \int_o^t (.)dr$$

$$L_2^{-1} = e^{-t}\int_o^t e^{r}(\cdot)dr$$

$$L_3^{-1} = \int_o^t (\cdot)dr$$

and $Y_1(t)$, $Y_2(t)$ and $Y_3(t)$ by the rule of coefficient ergodicity

$$Y_1(t) = 1$$

$$Y_2(t) = e^{-2t}$$

$$Y_3(t) = 1$$

Therefore, we have from (40) – (42)

$$u_m(t) = \chi_m u_{m-1}(t) + h\int_o^t R_m(\vec{u}_{m-1})dr + C_1 \quad \ldots \quad (43)$$

$$v_m(t) = \chi_m v_{m-1}(t) + he^{-t}\int_o^t e^{-r} R_m(\vec{v}_{m-1})dr + C_2 e^{-t} \quad \ldots \quad (44)$$

$$w_m(t) = \chi_m w_{m-1}(t) + h\int_o^t R_m(\vec{w}_{m-1})dr + C_3 \quad \ldots \quad (45)$$

Where

$$R_m(\vec{u}_{m-1}) = u'_{m-1}(t) + R_o \sum_{k=o}^{m-1} u_{m-1-k}(t)\ v_k(t) \quad \ldots \quad (46)$$

$$R_m(\vec{v}_{m-1}) = v'_{m-1}(t) + R_o \sum_{k=o}^{m-1} u_{m-1-k}(t) v_k(t) + v_{m-1}(t) \quad \ldots \quad (47)$$

$$R_m(\vec{w}_{m-1}) = w'_{m-1}(t) - v_{m-1}(t) \quad \ldots \quad (48)$$

The mth-order approximation of U(t), V(t) and W(t) are given by

$$u(t) \approx \sum_{n=o}^{m} u_n(t) \quad \ldots \quad (49)$$

$$v(t) \approx \sum_{n=o}^{m} v_m(t) \quad \ldots \quad (50)$$

$$w(t) \approx \sum_{n=o}^{m} w_m(t) \quad \ldots \quad (51)$$

We have

$$U_o(t) = 0.9$$

$$U_1(t) = 0.9 + h[e^{-t} R_o(0.01 - 0.01e^{t} + 0.1t) - t] - t$$

$$V_o(t) = 0.1e^{-t}$$

$$V_1(t) = e^{-4t}\left[e^{3+}(0.1 - 0.02hR_o) + e^{t} hR_o(0.02 - 0.05t)\right]$$

4.0 Analysis of the Result

We have the valid regions of h after plotting the h-curves of $u(t), v(t)$ and $w(t)$ as $-1.6 < h < -0.5$ for $u(t)$, $-1.6 < h < -0.1$ for $v(t)$ and $-1.5 < h < 0.2$ for $w(t)$

for $R_0 = 0.5$ and $R_0 = 2$.

Choosing a common valid region from the valid regions of $u(t), v(t)$ and $w(t)$. We have our common valid region as $-1.5 < h < -0.5$. Within this region $-1.5 < h < -0.5$, we can choose any value for h for convergence.

For the different values of h, We have the following plots for 10^{th} order approximation of $u(t)$, $v(t), w(t)$ when $R_0 = 2$

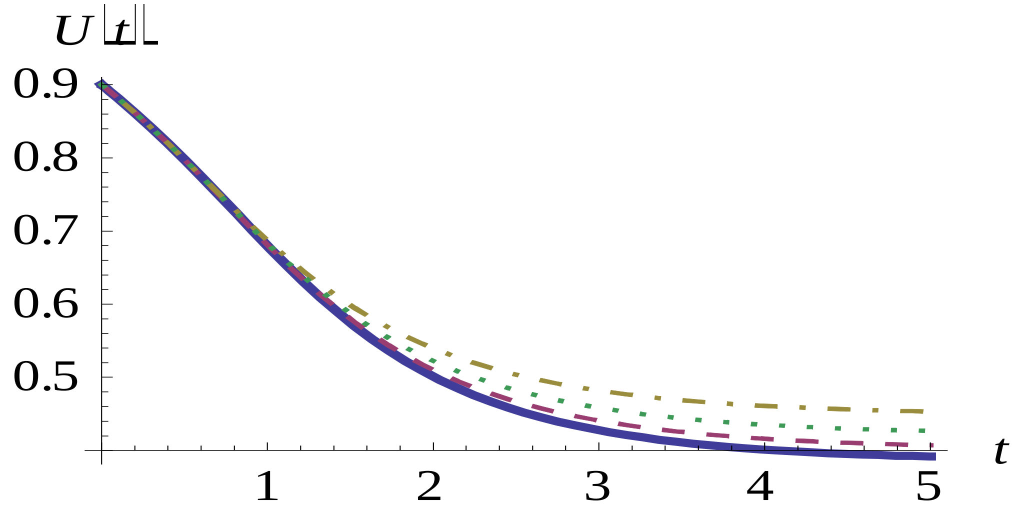

Fig.1. 10^{th} order approximation of $U(t)$ for $h = -0.8 \rightarrow$ dot dashed, $h = -1 \rightarrow$ dotted $h = -1.2 \rightarrow$ dashed and $h = -1.4 \rightarrow$ bold for $R_0 = 2$.

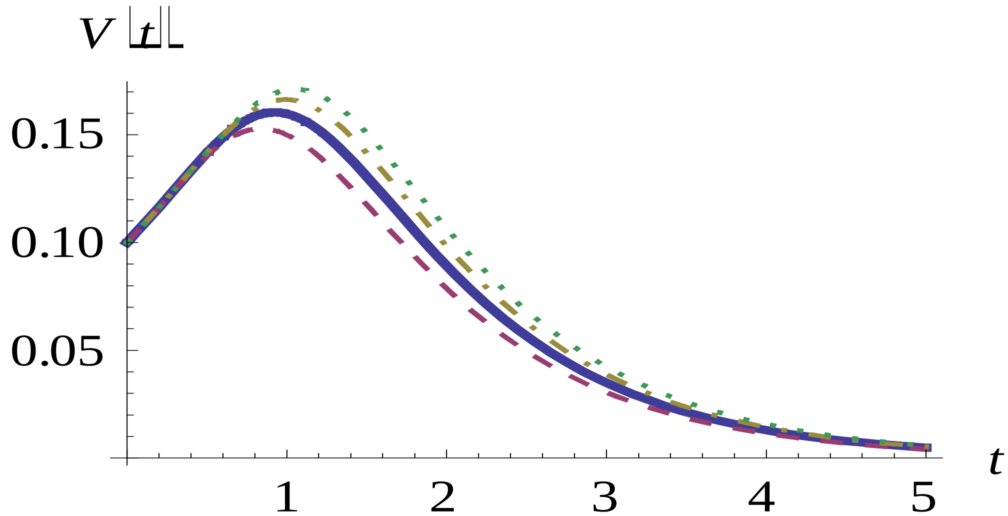

Fig.2. 10^{th} order approximation of $V(t)$ for $h = -0.8 \rightarrow$ dotted, $h = -1 \rightarrow$ Bold, $h = -1.2 \rightarrow$ dot dashed and $h = -1.4 \rightarrow$ dashed for $R_0 = 2$.

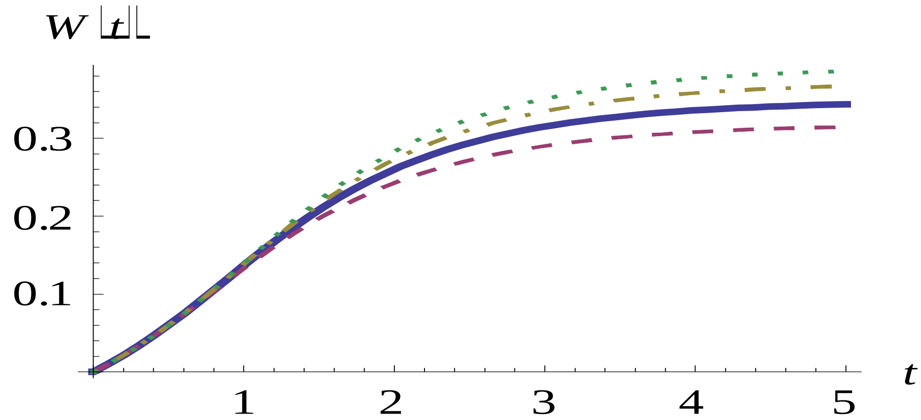

Fig.3. 10^{th} order approximation of $W(t)$ for $h=-0.8\rightarrow$ dotted, $h=-1\rightarrow$ Bold $h=-1.2\rightarrow$ dot dashed and $h=-1.4\rightarrow$ dashed for $R_0=2$.

For the difference value of h, We have the following plots for 10^{th} order approximation of $U(t), V(t)$, $W(t)$ when $R_0=0.5$. We have

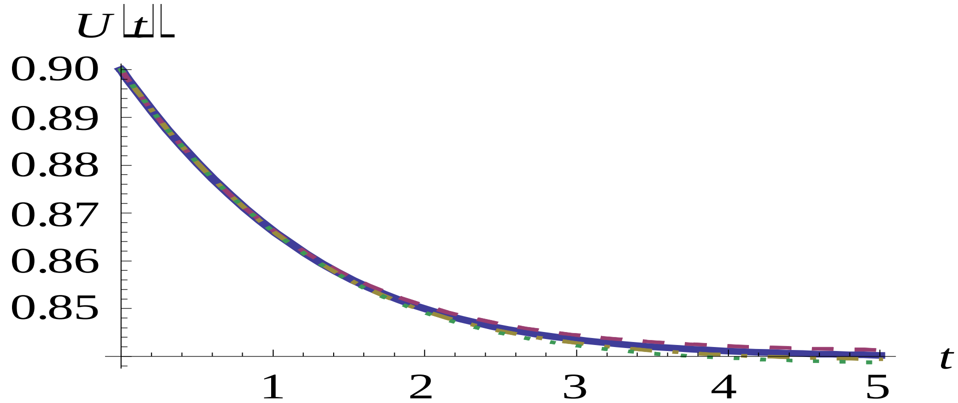

Fig.4. 10^{th} order approximation of $U(t)$ for $h=-0.8\rightarrow$ dotted, $h=-1\rightarrow$ Bold, $h=-1.2\rightarrow$ dot dashed and $h=-1.4\rightarrow$ dashed, $R_0=0.5$

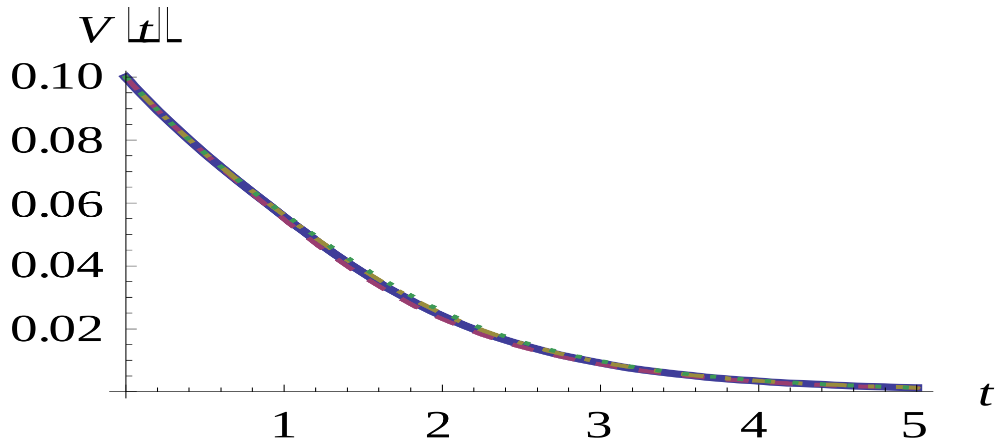

Fig.5. 10^{th} order approximation of $V(t)$ for $h=-0.8\rightarrow$ dotted, $h=-1\rightarrow$ Bold, $h=-1.2\rightarrow$ dot dashed and $h=-1.4\rightarrow$ dashed, $R_0=0.5$

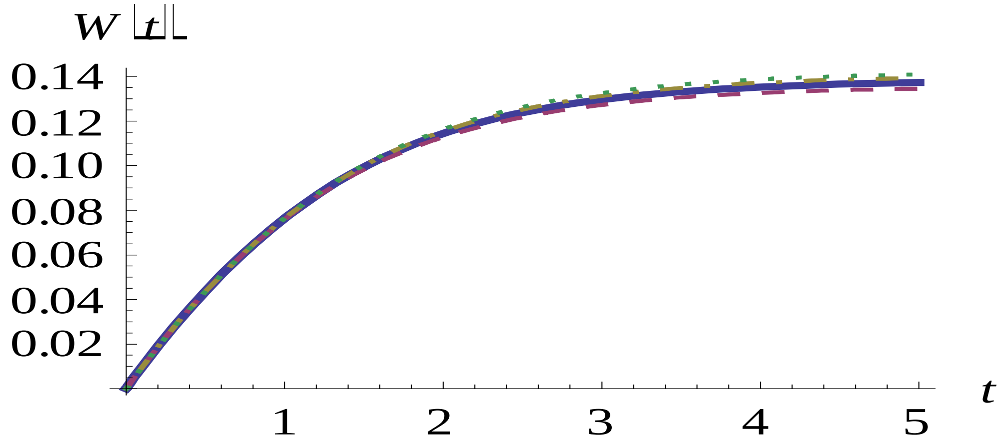

Fig.6. 10th order approximation of $W(t)$ for $h = -0.8 \rightarrow$ dotted, $h = -1 \rightarrow$ Bold $h = -1.2 \rightarrow$ dot dashed and $h = -1.4 \rightarrow$ dashed, $R_0 = 0.5$

5.0 Conclusion

From Fig 1, Fig 3, Fig. 4 and Fig. 6, the shapes of $U(t)$,and $W(t)$ follow the normal shape found by other authors [15 - 17] but we are more interested in the value of an epidemic hence our concentration is on Figs 2 and 5 for $V(t)$; which measures the size of an epidemic. We found out that when the basic reproduction number R_0 is less than 1 (Fig. 5), there is no epidemic, the number of infected persons decreases with time and tends to constant value. However when R_0 is greater than 1 (Fig 2), there would be an epidemic as predicted in literature [15-17]. From Fig. 6, the number of infected persons increases to a maximum and finally reduces.

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