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Convergence and Stability Analysis of a Symmetric Implicit Runge-Kutta Method for Direct Integration of First, Second and Third Order ODEs.

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Abstract

In this work, we present the convergence and stability analysis of an implicit 5-stage Symmetric Runge-Kutta method for direct integration of first, second and third order ODEs. In the process we plot the region of absolute stability, find the order and error constant, test for consistency and convergence of the method. Comparison of the convergence and stability analysis were also made.

Keywords: Symmetric Implicit Runge-Kutta Method, Region of Absolute Stability, Order and Error constant,Consistency and Convergence.

1.0 Introduction

A basic property which we shall demand of an acceptable numerical method is that the solution (y_n) generated by the method converges and stable in some sense, to the theoretical solution $(y(x))$ as the step length, n tends to zero. Stability properties of numerical methods are important for achieving a good approximation to the true solution. When a numerical method is used, there are often differences between the exact solution and the numerical solution at the mesh points; this is the local truncation error. Sometimes the accumulation of errors will cause instability. Therefore a method must satisfy the stability condition so that the numerical method converges to the exact solution [1].
The 5-stage symmetric implicit Runge-Kutta method(SIRK) of

$$y' = f(x,y), \qquad y(x_0) = y_0 \tag{1.1}$$

as presented earlier in [2] is

$$y_{n+1} = y_n + h(\frac{7}{90}k_1 + \frac{16}{45}k_2 + \frac{2}{15}k_3 + \frac{16}{45}k_4 + \frac{7}{90}k_5) \tag{1.2}$$

where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{1}{4}h, y_n + h(\frac{251}{2880}k_1 + \frac{323}{1440}k_2 - \frac{11}{120}k_3 + \frac{53}{1440}k_4 - \frac{19}{2880}k_5))$$

$$k_3 = f(x_n + \frac{1}{2}h, y_n + h(\frac{29}{360}k_1 + \frac{31}{90}k_2 + \frac{1}{15}k_3 + \frac{1}{90}k_4 - \frac{1}{360}k_5))$$

$$k_4 = f(x_n + \frac{3}{4}h, y_n + h(\frac{27}{320}k_1 + \frac{51}{160}k_2 + \frac{9}{40}k_3 + \frac{21}{160}k_4 - \frac{3}{320}k_5))$$

$$k_5 = f(x_n + h, y_n + h(\frac{7}{90}k_1 + \frac{16}{45}k_2 + \frac{2}{15}k_3 + \frac{16}{45}k_4 + \frac{7}{90}k_5))$$

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The method (1.2) was reformulated in [2] into 5-stage Symmetric Implicit Runge-Kutta Nyström(SIRKN) method for the solution of initial value problems of the form

$$y'' = f(x,y,y') \qquad y(x_0)=y_0,\ y'(x_0)=y'_0 \tag{1.3}$$

to have

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + h^2\left(\frac{7}{90}k_1 + \frac{4}{15}k_2 + \frac{1}{15}k_3 + \frac{4}{45}k_4 + 0k_5\right) \\ y'_{n+1} &= y'_n + h\left(\frac{7}{90}k_1 + \frac{16}{45}k_2 + \frac{2}{15}k_3 + \frac{16}{45}k_4 + \frac{7}{90}k_5\right) \end{aligned} \tag{1.4}$$

where

$$k_1 = f(x_n, y_n, y'_n)$$

$$k_2 = f\Big(x_n + \frac{1}{4}h,\ y_n + \frac{1}{4}hy'_n + h^2\left(\frac{17}{1152}k_1 + \frac{9}{320}k_2 - \frac{37}{1920}k_3 + \frac{7}{720}k_4 - \frac{1}{480}k_5\right),$$
$$y'_n + h\left(\frac{251}{2880}k_1 + \frac{323}{1440}k_2 - \frac{11}{120}k_3 + \frac{53}{1440}k_4 - \frac{19}{2880}k_5\right)\Big)$$

$$k_3 = f\Big(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}hy'_n + h^2\left(\frac{13}{360}k_1 + \frac{37}{360}k_2 - \frac{1}{40}k_3 + \frac{1}{72}k_4 - \frac{1}{360}k_5\right),$$
$$y'_n + h\left(\frac{29}{360}k_1 + \frac{31}{90}k_2 + \frac{1}{15}k_3 + \frac{1}{90}k_4 - \frac{1}{360}k_5\right)\Big)$$

$$k_4 = f\Big(x_n + \frac{3}{4}h,\ y_n + \frac{3}{4}hy'_n + h^2\left(\frac{9}{160}k_1 + \frac{3}{16}k_2 + \frac{9}{640}k_3 + \frac{9}{320}k_4 - \frac{3}{640}k_5\right),$$
$$y'_n + h\left(\frac{27}{320}k_1 + \frac{51}{160}k_2 + \frac{9}{40}k_3 + \frac{21}{160}k_4 - \frac{3}{320}k_5\right)\Big)$$

$$k_5 = f\Big(x_n + h,\ y_n + hy'_n + h^2\left(\frac{7}{90}k_1 + \frac{4}{15}k_2 + \frac{1}{15}k_3 + \frac{4}{45}k_4 + 0k_5\right),$$
$$y'_n + h\left(\frac{7}{90}k_1 + \frac{16}{45}k_2 + \frac{2}{15}k_3 + \frac{16}{45}k_4 + \frac{7}{90}k_5\right)\Big)$$

The method(1.4) was extended in [3] to a 5-stage Symmetric Implicit Super Runge-Kutta(SISRKN) method for Direct Integration Of Third Order initial value problems(IVPs) of the form

$$y''' = f(x,y) \qquad y(x_o)=y \qquad y'(x_o)=\beta \qquad y''(x_o)=\alpha \tag{1.5}$$

$$y''' = f(x,y,y',y'') \qquad y(x_o)=y \qquad y'(x_o)=\beta \qquad y''(x_o)=\alpha \tag{1.6}$$

as

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + \frac{(h)^2}{2}y''_n + h^3\left(\frac{13}{360}k_1 + \frac{1}{9}k_2 + 0k_3 + \frac{1}{45}k_4 - \frac{1}{320}k_5\right) \\ y'_{n+1} &= y'_n + hy''_n + h^2\left(\frac{7}{90}k_1 + \frac{4}{15}k_2 + \frac{1}{15}k_3 + \frac{4}{45}k_4 + 0k_5\right) \\ y''_{n+1} &= y''_n + h\left(\frac{7}{90}k_1 + \frac{16}{45}k_2 + \frac{2}{15}k_3 + \frac{16}{45}k_4 + \frac{7}{90}k_5\right) \end{aligned} \tag{1.7}$$

Where

$$k_1 = f(x_n, y_n, y'_n, y''_n)$$

$$k_2 = f(x_n + \frac{1}{4}h, y_n + \frac{1}{4}hy'_n + \frac{(\frac{1}{4}h)^2}{2}y''_n + h^3(\frac{287}{184320}k_1 + \frac{187}{92160}k_2 - \frac{1}{512}k_3 + \frac{25}{18432}k_4 - \frac{71}{184320}k_5),$$
$$y'_n + \frac{1}{4}hy''_n + h^2(\frac{17}{1152}k_1 + \frac{9}{320}k_2 - \frac{37}{1920}k_3 + \frac{7}{720}k_4 - \frac{1}{480}k_5),$$
$$y''_n + h(\frac{251}{2880}k_1 + \frac{323}{1440}k_2 - \frac{11}{120}k_3 + \frac{53}{1440}k_4 - \frac{19}{2880}k_5))$$

$$k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hy'_n + \frac{(\frac{1}{2}h)^2}{2}y''_n + h^3(\frac{91}{11520}k_1 + \frac{103}{5760}k_2 - \frac{1}{120}k_3 + \frac{5}{1152}k_4 - \frac{11}{11520}k_5),$$
$$y'_n + \frac{1}{2}hy''_n + h^2(\frac{13}{360}k_1 + \frac{37}{360}k_2 - \frac{1}{40}k_3 + \frac{1}{72}k_4 - \frac{1}{360}k_5),$$
$$y''_n + h(\frac{29}{360}k_1 + \frac{31}{90}k_2 + \frac{1}{15}k_3 + \frac{1}{90}k_4 - \frac{1}{360}k_5))$$

$$k_4 = f(x_n + \frac{3}{4}h, y_n + \frac{3}{4}hy'_n + \frac{(\frac{3}{4}h)^2}{2}y''_n + h^3(\frac{399}{20480}k_1 + \frac{111}{2048}k_2 - \frac{27}{2560}k_3 + \frac{93}{10240}k_4 - \frac{39}{20480}k_5),$$
$$y'_n + \frac{3}{4}hy''_n + h^2(\frac{9}{160}k_1 + \frac{3}{16}k_2 + \frac{9}{640}k_3 + \frac{9}{320}k_4 - \frac{3}{640}k_5),$$
$$y''_n + h(\frac{27}{320}k_1 + \frac{51}{160}k_2 + \frac{9}{40}k_3 + \frac{21}{160}k_4 - \frac{3}{320}k_5))$$

$$k_5 = f(x_n + h, y_n + hy'_n + \frac{(h)^2}{2}y''_n + h^3(\frac{13}{360}k_1 + \frac{1}{9}k_2 + 0k_3 + \frac{1}{45}k_4 - \frac{1}{320}k_5),$$
$$y'_n + hy''_n + h^2(\frac{7}{90}k_1 + \frac{4}{15}k_2 + \frac{1}{15}k_3 + \frac{4}{45}k_4 + 0k_5),$$
$$y''_n + h(\frac{7}{90}k_1 + \frac{16}{45}k_2 + \frac{2}{15}k_3 + \frac{16}{45}k_4 + \frac{7}{90}k_5))$$

As was stated in [4] , there are three ways of deriving and analyzing Runge –Kutta methods:

- Taylor series expansion
- The algebraic concept of rooted trees and
- Computer algebra

For this work, we are going to analyze our methods using the Taylor series expansion because it enables us to obtain the error constant of the methods.

Definition 1.1 Convergences

If $f(x, y(x))$, $f(x, y(x), y'(x))$ and $f(x, y(x), y'(x), y''(x))$ satisfies Lipschitz condition then for such method consistency is necessary and sufficient for convergence. Therefore the method (1.2),(1.4) and (1.7) are said to be convergent if and only if they are consistent (See [1]).

Definition 1.2: A-stable

A general linear method is 'A stable' if $M(z)$ is power bounded for every z in the left half complex plane.

Note that there is no requirement corresponding to zero-stability, since no parasitic solution can arise with Runge-Kutta method [1].

The work is organized as follows in section 2,3 and 4 we present the Convergence and stability analysis of a 5-stage Symmetric Implicit Runge-Kutta method for first,second and third order ODEs equation (1.2),(1.4) and (1.7) respectively. The conclusion (section 5) are drawn based on the result obtained in the above sections.

2.0 Convergence and stability analysis of a 5-stage Symmetric Implicit Runge-Kutta method for first order ODEs equation (1.2)

$$y_{n+1} = y_n + h(\frac{7}{90}k_1 + \frac{16}{45}k_2 + \frac{2}{15}k_3 + \frac{16}{45}k_4 + \frac{7}{90}k_5)$$

Since $k_i = f_{c_i}$ implies $k_1 = f_{c_1}$, $k_2 = f_{c_2}, k_3 = f_{c_3}, ..., k_s = f_{c_s}$ [5].

From(1.2) $k_1 = f_0$, $k_2 = f_{\frac{1}{4}}$, $k_3 = f_{\frac{1}{2}}, k_4 = f_{\frac{3}{4}}, k_5 = f_1$

Using Taylor's series expansion,

$$y(n+1) = y(n+h) = y(n) + hy'(n) + \frac{h^2}{2}y''(n) + ... + \frac{h^s}{s!}y^s(n)$$

$$f(n+1) = f_1 = f(n+h) = y'(n) + hy''(n) + \frac{h^2}{2}y'''(n) + ... + \frac{h^{(s-1)}}{(s-1)!}y^{s+1}(n)$$

and substituting into $y_{n+1} = y_n + h(\frac{7}{90}k_1 + \frac{16}{45}k_2 + \frac{2}{15}k_3 + \frac{16}{45}k_4 + \frac{7}{90}k_5)$,

we obtain following results. The error constant is $\frac{1}{322560}$, the order is 6 and the method is consistent since

$k_i(x, y(x), 0) \equiv f_{c_i}(x, y(x))$ and $\sum_1^s b_s = \frac{7}{90} + \frac{16}{45} + \frac{2}{15} + \frac{16}{45} + \frac{7}{90} = 1$, hence convergent (see definition (1.1) [1],[6].)

Matrices for stability polynomial [7] ,[8].

From equation (1.2) $M = \left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right]$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{251}{2880} & \frac{323}{1440} & \frac{-11}{120} & \frac{53}{1440} & \frac{-19}{2880} \\ \frac{29}{360} & \frac{31}{90} & \frac{1}{15} & \frac{1}{90} & \frac{-1}{360} \\ \frac{27}{320} & \frac{51}{160} & \frac{9}{40} & \frac{21}{160} & \frac{-3}{320} \\ \frac{7}{90} & \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{7}{90} & \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Using maple software gives the characteristic polynomial and stability function [9] as

$$M(z) = V + zB(1 - zA)^{-1}U$$

$$\frac{60\,(-245760 - 124\,z^4 + 7680\,z^2 + 5\,z^6)\,\eta}{(3\,z^4 - 50\,z^3 + 420\,z^2 - 1920\,z + 3840)^2}$$

$$\phi(\eta, z) = \det(\eta I - m(z)) =$$

$$\eta\,(3\,\eta\,z^4 - 50\,\eta\,z^3 + 420\,\eta\,z^2 - 1920\,\eta\,z + 3840\,\eta - 3\,z^4 - 50\,z^3 - 420\,z^2 - 1920\,z - 3840)\,/\,(3\,z^4 - 50\,z^3 + 420\,z^2 - 1920\,z + 3840)$$

Putting the characteristic polynomial and stability function in MATLAB software shows that the region absolute stability (RAS) of the method is A-stable.

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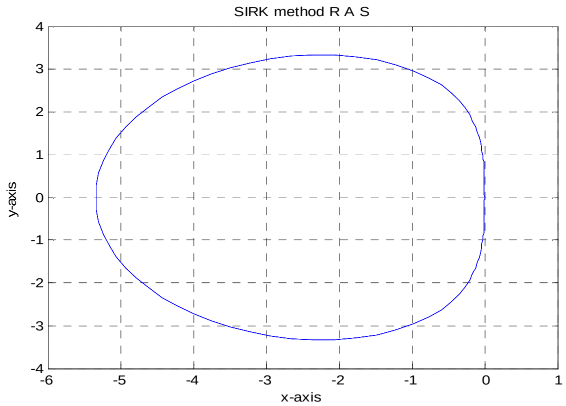

Figure 1.1: Stability plots for Symmetric Implicit Runge-Kutta(SIRK) for first order ODEs equation (1.2)

4.2 Convergence and stability analysis of a Symmetric Implicit Runge-Kutta Nyström method for second order ODEs(see equation (1.4))

$$y_{n+1} = y_n + hy_n' + h^2\left(\frac{7}{90}k_1 + \frac{4}{15}k_2 + \frac{1}{15}k_3 + \frac{4}{45}k_4 + 0k_5\right)$$

Using Taylor's series expansion

$$f(n+1) = f_1 = f(n+h) = y''(n) + hy'''(n) + \frac{h^2}{2}y''''(n) + \dots\dots\dots + \frac{h^{(s-2)}}{(s-2)!}y^{s+2}(n)$$

and substituting into $y_{n+1} = y_n + hy_n' + h^2\left(\frac{7}{90}k_1 + \frac{4}{15}k_2 + \frac{1}{15}k_3 + \frac{4}{45}k_4 + 0k_5\right)$,

we obtain following results. The error constant is $\frac{1}{322560}$, order is 5 and the method is consistent, since

$k_i(x, y(x), y'(x), 0) \equiv f_{c_i}(x, y(x), y'(x))$ and $\sum_1^s b_s = \frac{7}{90} + \frac{16}{45} + \frac{2}{15} + \frac{16}{45} + \frac{7}{90} = \frac{1}{2}$ [10] hence

convergent (see definition (1.1)), [1] , [6].

Matrices for stability polynomial [2], [3],[11] ,[12] are:

$$M = \left[\begin{array}{c|c|c} A & \bar{A} & U \\ B & \bar{B} & V \end{array}\right]$$

From equation(1.4),

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{251}{2880} & \frac{323}{1440} & \frac{-11}{120} & \frac{53}{1440} & \frac{-19}{2880} \\ \frac{29}{360} & \frac{31}{90} & \frac{1}{15} & \frac{1}{90} & \frac{-1}{360} \\ \frac{27}{320} & \frac{51}{160} & \frac{9}{40} & \frac{21}{160} & \frac{-3}{320} \\ \frac{7}{90} & \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{17}{1152} & \frac{9}{320} & \frac{-37}{1920} & \frac{7}{720} & \frac{-1}{480} \\ \frac{13}{360} & \frac{37}{360} & \frac{-1}{40} & \frac{1}{72} & \frac{-1}{360} \\ \frac{9}{160} & \frac{3}{16} & \frac{9}{640} & \frac{9}{320} & \frac{-3}{640} \\ \frac{7}{90} & \frac{4}{15} & \frac{1}{15} & \frac{4}{45} & 0 \end{bmatrix},$$

$$U=\begin{bmatrix}1&0\\1&0\\1&0\\1&0\\1&0\end{bmatrix},\quad B=\begin{bmatrix}\frac{7}{90}&\frac{16}{45}&\frac{2}{15}&\frac{16}{45}&\frac{7}{90}\\0&0&0&0&0\end{bmatrix},,\quad V=\begin{bmatrix}1&0\\1&0\end{bmatrix},\quad \overline{B}=\begin{bmatrix}\frac{7}{90}&\frac{4}{15}&\frac{1}{15}&\frac{4}{45}&0\\0&0&0&0&0\end{bmatrix}$$

Using maple software gives the characteristic polynomial and stability function as

$$M(z)=V+(zB+z^2\bar{B})(1-zA-z^2\bar{A})^{-1}U=$$

$$1800\,(-6291456000\,z+5898240\,z^2+2723840\,z^3+94192\,z^4+3840\,z^5+25\,z^6-60397977600\,)\,\eta\,/\,(9\,z^4+20\,z^3+7440\,z^2-460800\,z+14745600\,)^2$$

$$\phi(\eta,z)=\det(\eta I-m(z))=$$

$$\eta\,(9\,\eta\,z^4+20\,\eta\,z^3+7440\,\eta\,z^2-460800\,\eta\,z+14745600\,\eta-9\,z^4-5020\,z^3-391440\,z^2-6912000\,z-14745600\,)\,/\,(9\,z^4+20\,z^3+7440\,z^2-460800\,z+14745600\,)$$

Putting the characteristic polynomial and stability function in MATLAB software shows that the region absolute stability of the method is A-stable.

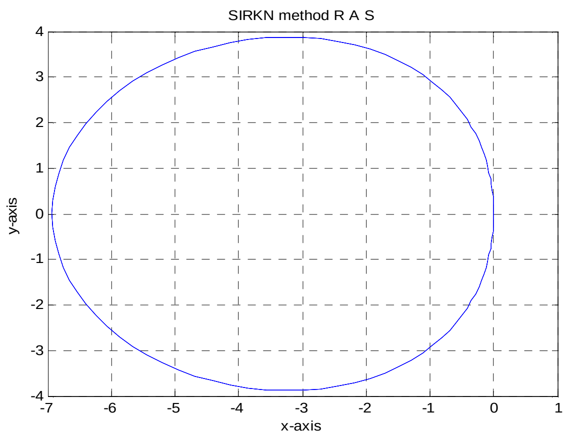

Figure 1.2: Stability plots for Symmetric Implicit Runge-Kutta Nyström method for second order ODEs equation (1.4)

4.3 Convergence and stability analysis of Symmetric Implicit Super Runge-Kutta Nyström Method(SISRKN) for direct integration of third order ODEs (see equation (1.7))

$$y_{n+1}=y_n+hy_n'+\frac{(h)^2}{2}y_n''+h^3\left(\frac{13}{360}k_1+\frac{1}{9}k_2+0k_3+\frac{1}{45}k_4-\frac{1}{320}k_5\right)$$

Using Taylor's series expansion

$$f(n+1)=f_1=f(n+h)=y'''(n)+hy''''(n)+\frac{h^2}{2}y'''''(n)+\ldots\ldots\ldots+\frac{h^{(s-3)}}{(s-3)!}y^{s+3}(n)$$

and substituting into $y_{n+1}=y_n+hy_n'+\frac{(h)^2}{2}y_n''+h^3(\frac{13}{360}k_1+\frac{1}{9}k_2+0k_3+\frac{1}{45}k_4-\frac{1}{320}k_5)$

we obtain following results. The error constant is $\frac{1}{322560}$, order is 4 and the method is consistent, since

$k_i(x,y(x),y'(x),y''(x),0)\equiv f_{c_i}(x,y(x),y'(x),y''(x))$ and $\sum_1^s b_s=\frac{7}{90}+\frac{16}{45}+\frac{2}{15}+\frac{16}{45}+\frac{7}{90}=\frac{1}{6}$ hence convergent (see definition (1.1)} ,[1] and [6]. Matrices for stability polynomial [3] are:

From equation (1.7), $M = \left[\begin{array}{c|c|c|c} A & \bar{A} & \bar{\bar{A}} & U \\ \hline B & \bar{B} & \bar{\bar{B}} & V \end{array}\right]$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{251}{2880} & \frac{323}{1440} & \frac{-11}{120} & \frac{53}{1440} & \frac{-19}{2880} \\ \frac{29}{360} & \frac{31}{90} & \frac{1}{15} & \frac{1}{90} & \frac{-1}{360} \\ \frac{27}{320} & \frac{51}{160} & \frac{9}{40} & \frac{21}{160} & \frac{-3}{320} \\ \frac{7}{90} & \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{17}{1152} & \frac{9}{320} & \frac{-37}{1920} & \frac{7}{720} & \frac{-1}{480} \\ \frac{13}{360} & \frac{37}{360} & \frac{-1}{40} & \frac{1}{72} & \frac{-1}{360} \\ \frac{9}{160} & \frac{3}{16} & \frac{9}{640} & \frac{9}{320} & \frac{-3}{640} \\ \frac{7}{90} & \frac{4}{15} & \frac{1}{15} & \frac{4}{45} & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{7}{90} & \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \frac{7}{90} & \frac{4}{15} & \frac{1}{15} & \frac{4}{45} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

$$\bar{\bar{A}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{287}{184320} & \frac{187}{92160} & \frac{-1}{512} & \frac{25}{18432} & \frac{-71}{184320} \\ \frac{91}{11520} & \frac{103}{5760} & \frac{-1}{120} & \frac{5}{1152} & \frac{-11}{11520} \\ \frac{399}{20480} & \frac{111}{2048} & \frac{-27}{2560} & \frac{93}{10240} & \frac{-39}{20480} \\ \frac{13}{360} & \frac{1}{9} & 0 & \frac{1}{45} & \frac{-1}{360} \end{bmatrix}, \quad \bar{\bar{B}} = \begin{bmatrix} \frac{13}{360} & \frac{1}{9} & 0 & \frac{1}{45} & \frac{-1}{360} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using maple software gives the characteristic polynomial and stability function as

$$M(z) = V + (zB + z^2\bar{B} + z^3\bar{\bar{B}})\left(1 - zA - z^2\bar{A} - z^3\bar{\bar{A}}\right)^{-1}U =$$

$$480\,(-18554258718720000\ z - 44505759744000\ z^2 + 478150656000\ z^3 + 3389790400\ z^4 + 8919\ z^6 + 8847360\ z^5 - 1113255523123200000\ )\,\eta \Big/ (27\ z^4 + 12160\ z^3 + 590400\ z^2 + 56623104000\ )^2$$

$$\phi(\eta, z) = \det(\eta I - m(z)) =$$

$$\eta\,(27\,\eta\, z^4 + 12160\,\eta\, z^3 + 590400\,\eta\, z^2 + 56623104000\ \eta - 27\, z^4 - 170720\, z^3 - 79233600\ z^2 - 56623104000\ - 9437184000\ z) \Big/ (27\ z^4 + 12160\ z^3 + 590400\ z^2 + 56623104000\ )$$

Putting the characteristic polynomial and stability function in MATLAB software shows that the region absolute stability of the method is A-stable.

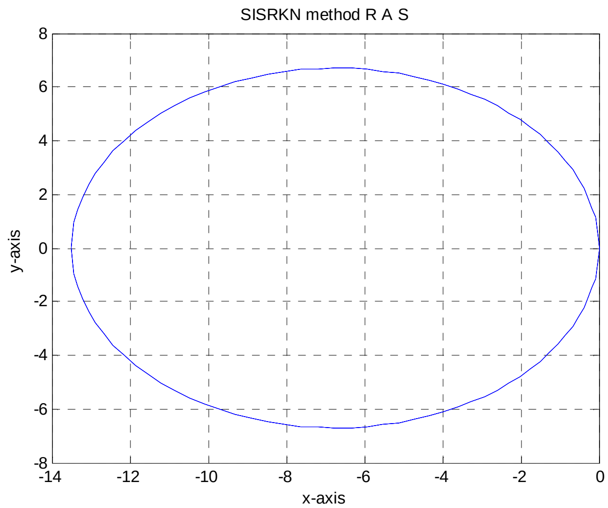

Figure 1.3: Stability plots for Symmetric Implicit Super Runge-Kutta Nyström Method (SISRKN) for direct integration of third order ODEs equation (1.7)

1.0 Conclusion

From sections 2, 3 and 4, the convergence and stability analysis of the 6-stage block implicit Runge-Kutta methods (1.2),(1.5)and (1.9) shows that, an order P method for a G(N) order ODEs extended for a higher order G(N+1) ODEs has order P-1, with increase inregion of absolute stability and same error constant leading same accuracy. Where P and N are integers and G (1),G(2),….,G(N) denote first order, second order, …nth order.

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