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Stochastic Decomposition in Non- Work Conserving Queues with Multiple Input Stream

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Abstract

Queues in which the server takes vacations or breaks down arise naturally as models for a wide range of computer, communication and production systems. We consider stochastic decomposition in non-work conserving queue with two independent input streams. One of the inputs to the queue is an independent and identically distributed process whereas the other is a general process and it is not required to be Markov nor is it required to be stationary. Time is divided into slots and the service interruption process is general. We show that the virtual waiting time in each slot in the queue is conditionally decomposed into two independent terms.

Keywords: Stochastic decomposition, Queues, Vacations, Input stream.

1.0 Introduction

Queuing theory involves the mathematical study of waiting lines. A queue system is a system consisting of flow of customers requiring services where there are some restrictions in the services that can be provided [1 - 10]. We can identify three main elements of a service centre: a population of customers, the service facility and the waiting line. Queuing theory tries to answer questions like, the mean waiting time in the queue, the mean waiting time in the queue, the mean response time (waiting time in the queue plus service time), mean utilization of service facilities distribution of the number of customer in the queue and so forth. These questions are mainly investigated in a stochastic scenario, where for example, the inter-arrival times of the customers or service times are assumed to be random. Waiting systems that admit interruptions of service often appear when the server uses idle periods of time of one queue or one task to serve clients in another queue or to perform another task. What matters is that, for these idle periods, the server is not available or operational for new arrivals to the system. Among other applications, these waiting systems appear as models for computer networks, telecommunications, production and quality control. The study of queuing systems with service interruptions has received a significant amount of attention of the researcher in the field. One type of service interruption has already been considered in the context of vacation queues where interruptions only happen as soon as the queue becomes empty. These vacation models are shown to exhibit a stochastic decomposition property. The stationary number of customers in the system can be interpreted as the sum of the state of corresponding system with no vacations and another non negative discrete random variable. A corresponding decomposition result occurs for the waiting time distribution as well.

A study on stochastic decomposition in the M/G/1 queue with generalized variations was carried out by Fuhrmann and Cooper [10]. They considered a class of M/G/1 queue models with a server who is unavailable for occasional intervals of time. As has been noted by other researchers [1, 2, 3, 4, 5, 9], for several specific model of this type, the stationary number of customers present in the system at a random point in time is distributed as the sum of two or more independent random variables, one of which is the stationary number of customers present in the standard M/G/1 queue (i.e. the server is always available) at a random point in time. Ishizaki, et. al [6] considered a discrete time queue with gated priority. The system consists of two queues and a gate. Ordinary customers arrive at the first queue at the gate in batches according to a batch Bernoulli process (BBP). When the gate opens, all ordinary customers at the first queue move to the second queue at a single server.

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It is assumed that intervals between successive openings of the gate are independent and identically distributed (i.i.d) and bounded. Furthermore, the travel times from the first queue to the second queue are assumed to the zero. The gate closes immediately after all the ordinary customers who are waiting in the first queue move to the second queue. In addition, there are also priority customers who directly join the second queue upon arrival. The arrival process
of priority customers is assumed to be a BBP. The server only serves customers in the second queue. Service times of ordinary customers are i.i.d. according to a general distribution function. Also, service times of priority customers are i.i.d. according to a general distribution which may differ from that for ordinary priority customers. Since priority customers do not need to wait for the gate opening, they have priority over ordinary customers. Therefore, they called this system, a queue with gated priority. For this queue, they derived the probability generating function for the amount of work in the system, the waiting times and the length of high priority and low propriety (ordinary) customers under the assumption of bounded gate opening intervals. Other work conserving queues which have been studied include an M/G/1 queueing model in which the server must search for customers [7, 8].

In this work we will consider stochastic decomposition in non-work conserving queue with two independent input streams. We will study three queues; the first queue has two input streams which are represented by A- and B- processes. The server in the first queue is subject to break down and the availability of the server in the first queue is determined by a stochastic process. The second queue with independent and identically distributed (i.i.d) input stream has only one input stream which is stochastically identical to the B-process. The server in the second queue with i.i.d. input stream is always available. The Third queue has only one input steam which is represented by the $\hat{A}$-process constructed from the A- and δ-processes and the distribution of a delayed busy period in the second queue with i.i.d. input stream, the server in the third queue is also subject to break down and the availability of the server in the third queue is also determined by the δ-process. We focus on the virtual waiting-time process in the first queue, that in the second queue with i.i.d. input stream and that in the third queue which are denoted by $\{X_n\}_{n\in\mathbb{Z}_+}, \{U_n\}_{n\in\mathbb{Z}_+}$ and $\{\hat{X}_n\}_{n\in\mathbb{Z}_+}$ respectively.

It will be shown that the virtual waiting time in each of the slot in the first queue is (in a sense "conditional distribution") decomposed into two independent terms: The virtual waiting time in the second queue with i.i.d. input steam and a quantity which is closely related to the virtual waiting time in the slot in the third queue. It is assumed that all stochastic processes and random variables are defined on a common probability space $(\Omega, \mathcal{F}, P)$. Also, all random variables described in this section are P-integrable.

2.0 Virtual and Actual Waiting Time

The time interval during which the server is continuously busy is called a busy period. In other words, if the nth customer finds the server idle, a new busy period begins at time t_n. Let X_t denote the number of customers in the system a time t. The event $X_t = n$ is the event that at time t, the server is busy and there are $n-1$ customers waiting.
The virtual waiting time at time t is defined as the length of time a (virtual) customer who arrives at time t has to wait before starting service.
Let $\gamma(t)$ denote the stochastic process defined at each instant t as the time elapsing from t until the server completes serving customers entering the queue before t. If at instant t, the server is free, $\gamma(t) = 0$. Denote the arriving time of customers by $t_1, t_2, t_3,\cdots$. Then for $t_n < t < t_{n-1}$ The process $\gamma(t)$ is defined by

$$\gamma(t) = \begin{matrix} 0 & if \quad \gamma(t) \le t - t_n \\ \gamma(t) - (t - t_n) & if \quad \gamma(t) \ge t - t_n \end{matrix} \qquad (2.1)$$

for $t = t_n$ we have equality

$$\gamma(t_n + 0) = \gamma(t_n - 0) + \eta_t \qquad (2.2)$$

where η denotes the service time for customers arriving at time t_n. The process $\gamma(t)$ is called virtual waiting process. It is a Markov process. Physical situations in which the total time required to serve any specified group of customers is (nearly) independent of the order in which these customers are served, such a system, is called Work-conserving queue, work being identified as the amount of service time. In a work-conserving queue, the service time of a customer is not affected by the order in which customers are served.

3.0 The Pollaczek-Khinchin Transform Equation

Suppose the a service times of different customers are independently distributed with a random distribution function B(t) which is not necessarily of a particular simple mathematical form. The arrival process is a Poisson process with parameter λ. Next, the family of random variables v_i denotes the number of customer arrival in the ith service. Denote N(t) as the number of customers in the system at time t. Now we define

$$N_i = N(t_i)$$

And it is straightforward to establish the following recurrence equation

$$N_{i+1} = (N_i - 1)^+ + v_i$$

Where $(x)^+ := \max\{0, x\}$.

This equation means that the number of customers in the system t time t_{n+1} is given by the number at t_n minus the customer leaving the system plus newly arriving customers. Additionally, we assume that $N_0 = 0$. If the system is in steady state, the time dependency disappears in the long run and the random variables N_i and v_i converge to random variables N and V respectively. Taking limits yields

$$N = (N - 1)^+ + V \tag{3.1}$$

Let G_N(z) be the probability generating function of N. From the last equation and from convolution property of probability generating function, we have,

$$G_N(\mathrm{z}) = G_{(N-1)^+}(\mathrm{z})v(z) = G_{N-1}(\mathrm{z}) \cdot G_v(z) \tag{3.2}$$

From definition $G_N(\mathrm{z}) = \sum_{k=0}^{\infty} Pr(N = k)z^k$

Also, from the definition of probability generating function we have

$G_{(N-1)^+}(\mathrm{z}) = \sum_{k=0}^{\infty}[Pr(n - 1)^+ = i\,]z^k$

The random variable $(N - 1)^+$ take the value zero if and only if N = 0 or N = 1 holds. Thus, we have

Pr $[(N - 1)^+ = 0]$=Pr$[N = 0]$ + Pr $[N = 1]$

On the other hand, $(N - 1)^+$ takes the value $i(i > 0)$ if and only if $N = i + 1$,

thus,

$$Pr[(N - 1)^+ = 0] = Pr[N = i + 1]. \tag{3.3}$$

Collecting like terms yields

$$G_{(N-1)^+}(\mathrm{z}) = (\Pr[N = 0] + \Pr[N = 1])z^0 + \sum_{v=1}^{\infty}[\Pr(N = v + 1) = i\,]z^v$$

$$= \Pr[N = 0] + \frac{1}{z}\left(\Pr[N = 1]\,z + \sum_{v=2}^{\infty}[\Pr(N = v)\,]z^v\right)$$

$$= \Pr[N = 0] + \frac{1}{z}[G_N(\mathrm{z}) - \Pr[N = 0] \tag{3.4}$$

If we take ρ as the utilization of the server, then the server will in steady state be free with probability $1 - \rho$. Thus we have $Pr[N = 0] = 1 - \rho$, yielding

$$G_{(N-1)^+}(\mathrm{z}) = \sum_{v=0}^{\infty}[\Pr(v \text{ customers arrive during service time})\,]z^v$$

$$= \sum_{v=0}^{\infty}[\Pr(V = v)\,]z^v$$

In order to compute $Pr[V = v]$, we use the continuous analogue to the law of absolute probability by looking at all possible interval length, which yields

$$[\Pr(V = v)\,] = \int_0^{\infty} p(v, \lambda x) b(x) dx$$

where b(x) is the probability density function of service time distribution B. Now we collect all together and get

$$G_v(\mathrm{z}) = \sum_{\mathrm{i}=0}^{\infty}\int_0^{\infty} \mathrm{p}(\mathrm{i}, \lambda x)\, b(x)dx \cdot z^i$$

$$= \sum_{\mathrm{i}=0}^{\infty}\int_0^{\infty} \frac{\lambda x}{\mathrm{i}!}\mathrm{e}^{-\lambda x}\, b(x)dx \cdot z^i$$

$$= \int_0^{\infty} \mathrm{e}^{-\lambda x}\sum_{\mathrm{i}=0}^{\infty}\frac{\lambda x z^i}{\mathrm{i}!} b(x)dx$$

$$= \int_0^{\infty} \mathrm{e}^{-\lambda x}\mathrm{e}^{\lambda x z} b(x)dx$$

$$= \int_0^{\infty} \mathrm{e}^{-\lambda x(1-z)} b(x)dx$$

$$= L_B\big(\lambda(1 - z)\big) \tag{3.5}$$

where $L_B(s) = \int_0^{\infty} e^{-st} dB(t)$ is the Laplace-Stietjes transform of the service time. Now we can plug all our result back into the equation to arrive at

$$G_N(\mathrm{z}) = \frac{G_N(\mathrm{z}) + (1-\rho)(1-z)}{z} \cdot L_B\big(\lambda(1 - z)\big) \tag{3.6}$$

Which we can rearrange to get

$$G_N(\mathrm{z}) = L_B\big(\lambda(1-z)\big) \cdot \frac{(1-\rho)(1-\mathrm{z})}{L_B\big(\lambda(1-z)\big) - z} \qquad (3.6)$$

which is the Pollaczek-Khinchin Transform equation.

4.0 The Queues

First, the virtual waiting-time process in the first queue is described. The queue has two independent input streams. The server is subject to breakdown and the availability of the server is stochastically determined by $\{\delta_n\}_{n\in\mathbb{Z}_+}$ on$\{0,1\}$ called the δ-process. To describe the two input streams, introduced are two stochastic processes, $\{A_n\}_{n\in\mathbb{Z}_+}$and $\{B_n\}_{n\in\mathbb{Z}_+}$which represents the amount of work brought into the first queue in each slot. The processes $\{A_n\}$ and $\{B_n\}$ are called the A-process and B-process, respectively. Let X_n, $(n \in \mathbb{Z}_+)$ denote a random variables on $\mathbb{Z}_+$ representing the virtual waiting time in the first queue in the nth slot. The virtual waiting time process evolves from initial state X_0 according to the following recursion formular

$$X_{n+1} = (X_n - \delta_n)^+ + \mathrm{A}_n + \mathrm{B}_n \qquad (4.1)$$

A−and the δ-processes are not enquired to be Markov. Also, they are not required to be stationary or ergodic. A dependency between A- process and δ-process is allowed. For example, A_n may depend on $\{\delta_l\}_{l\le n}$ due to some control mechanism. Let an increasing sequence of sub-σ-field $\{\mathcal{F}_n\}_{n\in\mathbb{Z}_+}$and a sub-$\sigma$-field $\mathcal{F}_\infty$ of $\mathcal{F}$ be defined by

$\mathcal{F}_n = \sigma(A_0,\cdots,A_n,\delta_0,\cdots,\delta_n)$ and $\mathcal{F}_\infty = \sigma(A_n,\delta_n)$

The process $\{B_n\}_{n\in\mathbb{Z}_+}$is an i.i.d. sequence which is independent of $\sigma(X_0) \vee \mathcal{F}_\infty$. Furthermore, the traffic intensity of B-process is less than one. (That is, $\rho_B = E[B_n] < 1$). For any sub-σ-field $g \subset \mathcal{F}$, a generating function $X_n^*(\cdot\,|g)$ associated with the virtual waiting process in the first queue is defined by $X_n^*(z|g) = E[z^{X_n}|g]$. Also, a generating function, $B^*(\cdot)$ associated with the B-process is defined by

$$B^*(z) = E[z^{B_n}] \qquad (4.2).$$

Next, the virtual waiting-time process in the second queue with i.i.d. input stream, which is stochastically identical to the B-process is described. The server is always available. Let U_n denote the virtual waiting time in the stationary queue with i.i.d. input stream in the nth-slot. The process U_n evolves according to the following recursion

$$U_{n+1} = (U_n - 1)^+ + B_n \qquad (4.3)$$

That is, the number of customers in the system at instant t_{n+1} is given by the number of customers at time t_n minus the customer leaving, plus newly arriving customers. It is known, under stability condition, $\rho_B < 1$. It is assumed that $\{U_n\}$ is stationary. Let $U^*(\cdot)$ denote the generating function of the virtual waiting time U_n in the second queue with i.i.d. input stream. $U^*(z) = E[z^{U_n}]$. From the result in section 3.0 it can be seen that

$$U^*(z) = \frac{(z-1)B^*(z)}{z - B^*(z)}(1-\rho_B) \qquad (4.4)$$

Finally, the virtual waiting time process in the third queue is described. The server in the queue is subject to breakdown and the availability of the server is determined by the δ-process in the same manner as the first queue. This third queue has only one input stream, which is represented by the $\hat{A}$-process specified below. Let $\hat{X}$ denote a random variable on $\mathbb{Z}_+$ representing the virtual waiting time in the queue in the n-th slot. Then

$$\hat{X}_{n+1} = (\hat{X}_n - \delta_n)^+ + A_n. \qquad (4.5)$$

$\hat{X}_n^*(z|g) = E[z^{\hat{X}_n}|g])$ and $\hat{A}_n^*(z|g) = E[z^{\hat{A}_n}|g])$.

Let $\hat{\theta}$ denote a generic random variable representing a delayed busy period in the second queue with independent and identically distributed input stream. The generating function $\hat{\theta}$ is defined as

$$\theta^*(z) = E\big[z^{\hat{\theta}}\big] = B^*\big(z\theta^*(z)\big) \qquad (4.6)$$

Let $g_n = \sigma(A_n, \delta_n)$, with assumption that the conditional distribution of $\hat{A}$ is determined by the distribution of delayed busy period in the second queue with i.i.d input stream. That is

$$\hat{A}_n^*(z|g_n) = (z\theta^*(z))^{A_n}\big[\mathrm{I}_{\{\delta_n=0\}}\theta^*(z) + \mathrm{I}_{\{\delta_n=1\}}\big] \qquad (4.7)$$

Where $\mathrm{I}(\cdot)$ is the indicator function.

5.0 The Decomposition Result

To establish the decomposition result, we link the conditional distribution of the initial virtual waiting time in the third queue to that of the first queue.

$$X^*(z|\mathcal{F}_m) = U^*(z)\hat{X}_n(z/B^*(z)|\mathcal{F}_m) \qquad (5.1)$$

for any $n \in \{0,\ldots,m+1\}$

That is that the virtual waiting time in each of the slot in the first queue is (in a sense "conditional distribution") decomposed into two independent terms: The virtual waiting time in the second queue with i.i.d. input steam and a quantity which is closely related to the virtual waiting time in the slot in the third queue. We will show this by induction.

We allow a dependence between the conditional distribution of the initial virtual waiting time in the first queue and that in the third queue. The length of delay cycle starting with X_0 (i.e. the initial virtual waiting time in the original queue) in the second queue with independent and identically distributed input stream is equal (in the sense in distribution conditioned by $\mathcal{F}_m$) to the sum of two independent components, one of which is the length of a delay cycle starting with U_0 (i.e. the virtual waiting time in the second queue with independent and identically distributed input stream) in the stationary queue stream and the other of which is $\hat{X}_0$.

That is if there exists a non-negative integer m, such that:
$X_0{}^*(z|\mathcal{F}_m) = U^*(z)\hat{X}_0(z/B^*(z)|\mathcal{F}_m)$.
For notational convenience, we define $\mathcal{F}_n^B = \sigma(B_0, \dots, B_n)$ and $\mathcal{F}_{-1}^B = \{\phi, \Omega\}$
$(X_n - \delta_n)^+ + \mathrm{A}_n$ is a function of $(X_0, A_0, \dots, A_n, \delta_0, \dots, \delta_n, \mathrm{B}_0, \dots, \mathrm{B}_{n-1})$. Thus for $n \in \mathbb{Z}_+$ with $n \le m$ $(X_n - \delta_n)^+ + \mathrm{A}_n$ is $\sigma(X_0) \vee \mathcal{F}_m \vee \mathcal{F}_{n-1}^B -$ measurable. Since B_n and $\sigma(X_0) \vee \mathcal{F}_m \vee \mathcal{F}_{n-1}^B$ are independent, thus for any $n \in \mathbb{Z}_+$ with $n \le m$

$$E[z^{X_{n+1}}|\mathcal{F}_m] = E[z^{(X_n-\delta_n)^+ + \mathrm{A}_n + \mathrm{B}_n}|\mathcal{F}_m]$$
$$= E[E[z^{(X_n-\delta_n)^+ + \mathrm{A}_n + \mathrm{B}_n}|\sigma(X_0) \vee \mathcal{F}_m \vee \mathcal{F}_{n-1}^B \mathcal{F}_m]|\,\mathcal{F}_m]$$
$$= E[z^{(X_n-\delta_n)^+ + \mathrm{A}_n} E[z^{B_n}]|\mathcal{F}_m]$$
$$= B^*(\mathrm{z})E\left[z^{(X_n-\delta_n)^+ + \mathrm{A}_n}\middle|\mathcal{F}_m\right] \qquad (5.2)$$

On the other hand for $n \in \mathbb{Z}_+$ with $n \le m$ since (A_n, δ_n) is $\mathcal{F}_m$-measurable,

$$E\left[z^{(X_n-\delta_n)^+ + \mathrm{A}_n}\middle|\mathcal{F}_m\right] = E[\mathrm{I}_{\{\delta_n=0\}} z^{(X_n+A_n)}|\mathcal{F}_m] + E[\mathrm{I}_{\{\delta_n=1\}} z^{(X_n-1)^+ + \mathrm{A}_n}|\mathcal{F}_m]$$
$$= \mathrm{I}_{\{\delta_n=0\}} z^{A_n} X_n^*(z|\mathcal{F}_m) + \mathrm{I}_{\{\delta_n=1\}} z^{A_n} \frac{1}{z}[X^*(z|\mathcal{F}_m) + (z-1)X_n^*(0|\mathcal{F}_m) \qquad (5.3)$$

From (5.1) and (5.2), we obtain

$$X_{n+1}^*(z|\mathcal{F}_m) = z^{A_n} B^*(\mathrm{z}) \left[X_n^*(z|\mathcal{F}_m)\mathrm{I}_{\{\delta_n=0\}} + \frac{X_n^*(z|\mathcal{F}_m)\mathrm{I}_{\{\delta_n=1\}}(z-1)X_n^*(0|\mathcal{F}_m)}{z} \mathrm{I}_{\{\delta_n=1\}} \right]$$

We define an increasing sequence of sub-σ-fields $\{\mathcal{F}_n^{\hat{A}}\}_{n\in\mathbb{Z}_+}$ of $\mathcal{F}$ by
$\mathcal{F}_n^{\hat{A}} = \sigma(\hat{A}_0, \dots, \hat{A}_0)$ and $\mathcal{F}_{-1}^{\hat{A}} = \{\phi, \Omega\}$.
$(\hat{X}_n - \delta_n)^+$ is $\sigma(\hat{X}_0) \vee \mathcal{F}_m \vee \mathcal{F}_{n-1}^{\hat{A}}$-measurable and noting that $\mathcal{F}_n \subset \mathcal{F}_m$ we obtain,

$$E\left[z^{\hat{X}_{n+1}}\middle|\mathcal{F}_m\right] = E\left[z^{(\hat{X}_n-\delta_n)^+ + \mathrm{A}_n}\middle|\mathcal{F}_m\right]$$
$$= E\left[E\left[z^{(\hat{X}_n-\delta_n)^+ + \hat{\mathrm{A}}_n}\middle|\sigma(\hat{X}_0) \vee \mathcal{F}_m \vee \mathcal{F}_{n-1}^{\hat{A}} \mathcal{F}_m\right]\middle|\mathcal{F}_m\right]$$
$$= E[z^{(\hat{X}_n-\delta_n)^+} E[z^{\hat{A}_n}|\sigma(\hat{X}_0) \vee \mathcal{F}_m \vee \mathcal{F}_{n-1}^{\hat{A}}]|\mathcal{F}_m]$$

Noting that $g_n \subset \mathcal{F}_m$

$$E\left[z^{\hat{A}_n}\middle|g_n\right] = E[z^{\hat{A}_n}|\sigma(\hat{X}_0) \vee \mathcal{F}_m \vee \mathcal{F}_{n-1}^{\hat{A}}]$$

We have
$$E\left[z^{\hat{X}_{n+1}}\middle|\mathcal{F}_m\right] = E\left[z^{(\hat{X}_n-\delta_n)^+} E\left[z^{\hat{A}_n}\middle|g_n\right]\middle|\mathcal{F}_m\right] = E[z^{(\hat{X}_n-\delta_n)^+}|\mathcal{F}_m]\,E\left[z^{\hat{A}_n}\middle|g_n\right]$$
Since δ_n is $\mathcal{F}_m$-measurable, the first term on the right hand side of the above equation becomes

$$E\left[z^{\hat{X}_{n+1}}\middle|\mathcal{F}_m\right] = E\left[\mathrm{I}_{\{\delta_n=0\}} z^{\hat{X}_n}\middle|\mathcal{F}_m\right] + E\left[\mathrm{I}_{\{\delta_n=1\}} z^{(\hat{X}_n-1)^+}\middle|\mathcal{F}_m\right]$$
$$= \mathrm{I}_{\{\delta_n=0\}} E\left[z^{\hat{X}_n}\middle|\mathcal{F}_m\right] + \mathrm{I}_{\{\delta_n=1\}} E\left[\left[z^{(\hat{X}_n-1)^+}\middle|\mathcal{F}_m\right]\right.$$
$$= \mathrm{I}_{\{\delta_n=0\}} \hat{X}_m^*(z|\mathcal{F}_m) + \mathrm{I}_{\{\delta_n=1\}} \frac{1}{z}[\hat{X}_n{}^*(z|\mathcal{F}_m) + (z-1)\hat{X}_n{}^*(0|\mathcal{F}_m)$$
$$\hat{X}_{n+1}^*(z|\mathcal{F}_m) = \hat{A}(z|g_n) \left[\hat{X}_n{}^*(z|\mathcal{F}_m)\mathrm{I}_{\{\delta_n=0\}} + \frac{\hat{X}_n{}^*(z|\mathcal{F}_m) + (z-1)\hat{X}_n{}^*(0|\mathcal{F}_m)}{z} \mathrm{I}_{\{\delta_n=1\}} \right]$$

Equation (5.1) holds when n=0, suppose it holds for some $n = k$ $(k \le m)$ then,

$$X^*(z|\mathcal{F}_m) = U^*(z)\hat{X}_k(z/B^*(z)|\mathcal{F}_m)$$

For n=k+1, we have

$$X_{k+1}^*(z|\mathcal{F}_m) = z^{A_k} B^*(z) \left[U^*(z)\hat{X}^*{}_k(z/B^*(z)|\mathcal{F}_m)\mathrm{I}_{\{\delta_k=0\}} \right.$$
$$\left. + \frac{U^*(z)\hat{X}^*{}_k(z/B^*(z))|\mathcal{F}_m) + (z-1)(1-\rho_B)\hat{X}_k{}^*(0|\mathcal{F}_m)}{z} \mathrm{I}_{\{\delta_k=1\}} \right]$$

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$$= z^{A_k}U^*(z)\left[B^*(z)\hat{X}^*{}_k(z/B^*(z)|\mathcal{F}_m)\mathrm{I}_{\{\delta_k=0\}} + \frac{B^*(z)\hat{X}^*{}_k(z/B^*(z)|\mathcal{F}_m) + \left(z - B^*(z)\right)\hat{X}_k{}^*(0|\mathcal{F}_m)}{z}\mathrm{I}_{\{\delta_k=1\}}\right] \quad (5.4)$$

$$\hat{X}^*{}_k(z/B^*(z)|\mathcal{F}_m)$$
$$= \hat{A}_k^*(z/B^*(z)|g_k)\left[\hat{X}^*{}_k(z/B^*(z)|\mathcal{F}_m)\mathrm{I}_{\{\delta_k=0\}} + \frac{B^*(z)\hat{X}^*{}_k(z/B^*(z)|\mathcal{F}_m) + (z - B^*(z))\hat{X}_k{}^*(0|\mathcal{F}_m)}{z}\mathrm{I}_{\{\delta_k=1\}}\right]$$
$$= z^{A_k}\left[\mathrm{I}_{\{\delta_k=0\}}B^*(z) + \mathrm{I}_{\{\delta_k=1\}}\right]\left[\hat{X}^*{}_k(z/B^*(z)|\mathcal{F}_m)\mathrm{I}_{\{\delta_k=0\}} + \frac{B^*(z)\hat{X}^*{}_k(z/B^*(z)|\mathcal{F}_m) + \left(z - B^*(z)\right)\hat{X}_k{}^*(0|\mathcal{F}_m)}{z}\mathrm{I}_{\{\delta_k=1\}}\right]$$
$$= z^{A_k}\left[B^*(z)\hat{X}^*{}_k(z/B^*(z)|\mathcal{F}_m)\mathrm{I}_{\{\delta_k=0\}} + \frac{B^*(z)\hat{X}^*{}_k(z/B^*(z)|\mathcal{F}_m) + \left(z - B^*(z)\right)\hat{X}_k{}^*(0|\mathcal{F}_m)}{z}\mathrm{I}_{\{\delta_k=1\}}\right] \ldots \ldots (5.5)$$

From (5.4) and (5.5) it is derived that for $k \leq m$

$$X_{k+1}^*(z|\mathcal{F}_m) = U^*(z)\hat{X}_{k+1}(z/B^*(z)|\mathcal{F}_m)$$

Conclusion

It has been shown that the virtual waiting time in each of the slot in the first queue is (in a sense "in conditional distribution") decomposed into two independent terms: The virtual waiting time in the second queue with i.i.d. input stream and a quantity which is closely related to the virtual waiting time in the slot in the third queue. It is assumed that all stochastic processes and random variables are defined on a common probability space $(\Omega, \mathcal{F}, P)$.

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