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Simulation Of Conservation Of Energy And Angular Momentum In Binary Stars

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Abstract

The energy and angular momentum conservative theorems were investigated in binary stars with a program "cluster" which uses the leapfrog process. During simulation, instability was observed in the conservation process such that for cluster stars, the energy stabilization does not conserve angular momentum. The program was validated by using it in comparison with NASA's published data, to accurately simulate orbits of the moving bodies in a solar system model.

1.0 Introduction

When viewed through a telescope, many stars appear in systems of two or three. Astronomers thought that observed double stars might be accidental pairings of nearby and distant stars in the same line of sight. However, continued observations reveal that few stars occur singly while those that exist as multiples, orbit about each other. The number of stars composed within multiple star systems, ranges from few stars to clusters of hundreds to tens of millions of stars [1]. The physical systems we wish to study are often very complex, containing entities that cannot be seen and whose existence and properties can only be inferred [2].

This work is one such examples of modelling that comes under particle simulation (N-body problems) where the number of bodies is small and there is one-to-one correlation between the bodies in the system and those in the simulation. In particle-interaction problem, it is necessary to consider interaction between all pairs of bodies. N-body problems can be solved by numerically integrating up the differential equations of motion. Many different ways to do this to varying degrees of accuracy and speed exist. The simplest integrator is the Euler method, but this is only first order. A second order method is the leapfrog integration.

Assuming that interactions are symmetrical, so that the force on star i due to star j, $F_{ij,}$ is equal to - F_{ij} and those stars do not exert forces on themselves, then the total number of pair interactions is N(N-1)/2, where N is the number of stars. Thus for a cluster of hundred stars, there are 4950 pair interactions which can easily be handled with computers of modest power.

Binary stars are a system of stars consisting of two stars; a brighter star and a less bright star. The brighter star is called the primary and the less bright its companion or the secondary [3]. The two stars can either be optical doubles with no physical connection whatsoever (that is, they just appear close together in the sky as seen from Earth,) or, they can be connected, with both stars orbiting around their common centre of mass, which is what this work considers.

Most of these binary systems are in pairs with an orbital period of less than about 10 days and these orbits are usually circular, ''aligned'' (i.e., the spin axes of the two stars and the orbital plane axis are parallel), and synchronized (i.e., each star completes a single rotation about its axis once per orbit, and thus each star always sees the same side of its companion

2.0 Center of Mass For Binary Stars

The binary star system is a two-body problem, which is a problem that determines the motion of two point particles that interact only with each other. For a 2-BODY system, at time $t_{k,}$ if there exists a star p_i of mass m_i at position $(x_{i,k}, y_{i,k})$ i=1,2 within a system having total mass $M = m_1 + m_2$, then there is a unique point (x_k, y_k) such that equations (1) and (2) below are the centre of mass of the system at time t_k [4].

$$Mx_k = m_1 x_{1,k} + m_2 x_{2,k} \quad (1)$$

$$My_k = m_1 y_{1,k} + m_2 y_{2,k} \quad (2)$$

To consider the motion of the centre of mass, let time$t_k = K\Delta t$, $k = 0,1,2$ ...

$$m_1 a_{1,k,x} + m_2 a_{2,k,x} = 0, \qquad k \geq 0 \quad (3)$$

$$m_1\left(v_{1,k+1,x} - v_{1,k,x}\right) + m_2\left(v_{2,k+1,x} - v_{2,k,x}\right) = 0 \quad (4)$$

where m_1, m_2 are the masses of individual stars, a_1, a_2 the accelerations of the stars, v_1, v_2 the velocities of the stars respectively.

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Summing both sides of equation (4), over k from j-1, where $j \geq 1$ yields

$$m_1 v_{1,j,x} + m_2 v_{2,j,x} = C_1, \quad j \geq 0 \tag{5}$$

where $C_1 = m_1 v_{1,0,x} + m_2 v_{2,0,x}$

then if equation (6) is valid for any j, it must be valid for j+1 such that

$$m_1 v_{1,j+1,x} + m_2 v_{2,j+1,x} = C_1$$

$$m_1 x_{1,n} + m_2 x_{2,n} = C_1 t_n + C_2, \quad n \geq 0 \tag{6}$$

where $C_2 = m_1 x_{1,0,} + m_2 x_{2,0,}$

3.0 The Leap-Frog Process

The leapfrog process is used within the computation to compute the positions and velocities of the stars within the cluster at interleaved time points (interleaved such that they leapfrog each other). It is not a self-starting process thus it requires the value of a previous time step [5].

The computation determines the derivative in time across a double time step and uses the intermediate time step to define the integral ''f'' at

$$n+1 \qquad u^{n+1} = u^{n-1} - f(u^n, t^n)2\Delta t \tag{7}$$

$$n+2 \qquad u^{n+2} = u^n - f(u^{n+1}, t^{n+1})2\Delta t \tag{8}$$

thus $u^{n+1}$and u^{n+2} may be used in turn to determine u^{n+3}. It is clearly very simple and second order accuracy is obtained.

The difficulty in this method is that the boundary conditions only defines $u^0 = u(0)$ and to proceed it requires $u' = u(\Delta t)$, which is then determined by the Euler method and then it is proceeded for all subsequent time levels by the leapfrog process. It turns out that the overall accuracy of the leapfrog process is a very sensitive function of the accuracy of u' [6].

4.0 Conservation Of Energy In Binary Stars

Considering both the attractive and repulsive forces, the general force equation for a two-body problem will thus be;

$$f_{12,k} = -\frac{Gm_1m_2(x_{k+1}+x_k)}{r_k r_{k+1}(r_k+r_{k+1})} + \frac{Hm_1m_2\left[\sum_{j=0}^{m-2}\left(r_k^j r_{k+1}^{m-j-2}\right)\right](x_{k+1}+x_k)}{r_k^{m-1} r_{k+1}^{m-1}(r_k+r_{k+1})} \tag{9}$$

where i= 1,2 $j = 1,2,3, \ldots$, H is a constant of repulsion and $n \geq 2$.

The same follows for the y and z components.

To show that energy is conserved we need only establish the potential function V_k since it is known that the attraction part of equation (9) is energy conserving.

Defining the repulsion potential term as

$$V_{k(repulsion)} = H\frac{m_1m_2}{r_k^{m-1}} \tag{10}$$

Then

$$W_{n(repulsion)} = v_{0(repulsion)} - v_{n(repulsion)} \tag{11}$$

and

$$V_{k(attraction)} = -G\frac{m_1m_2}{r_k^{n-1}} \tag{12}$$

$$W_{n(attraction)} = -(v_{0(attraction)} - v_{n(attraction)}) \tag{13}$$

To find the kinetic energy function, it is known that

$$w_{n(kinetic)} = \frac{M}{2}v_{n,x}^2 - \frac{M}{2}v_{0,x}^2 \tag{14}$$

with the corresponding y-component equation as follows

$$w_{n(kinetic)} = \frac{M}{2}v_{n,y}^2 - \frac{M}{2}v_{0,y}^2 \tag{15}$$

Combining (14) and (15) gives

$$w_{n(kinetic)} = \frac{M}{2}\left(v_{n,x}^2 + v_{n,y}^2\right) - \frac{M}{2}\left(v_{0,x}^2 + v_{0,y}^2\right) \tag{16}$$

If the kinetic energy is generally defined as

$$k_i = \frac{m}{2}|v_i|^2 = \frac{m}{2}\left(v_{ix}^2 + v_{iy}^2\right) \tag{17}$$

then

$$w_{n(kinetic)} = k_n - k_0 \tag{18}$$

For the whole system, the total energy can be written as

$$\frac{k_0}{2} + v_0 = \frac{k_n}{2} + v_n \tag{19}$$

Equation (19) is the law of conservation of energy for a system of two stars.

5.0 Conservation Of Angular Momentum

Angular momentum can be defined as a vector quantity which varies directly with a body's mass, its distance r from the centre, and with its speed [4]. It is most convenient, to consider the angular momentum of a collection of particles about their centre of mass. Thus the angular momentum of the particles will be the sum of the angular momentum of each particle.
At t_k, if star p of mass m at position r_k with velocity V_k then the star's angular momentum $\boldsymbol{L}(t_k) = \boldsymbol{L_k}$ is defined by

$$\boldsymbol{L_k} = m(\boldsymbol{r_k} \times \boldsymbol{V_k}) \quad (20)$$

If $|\boldsymbol{r_k}|$ and $|\boldsymbol{V_k}|$ do not vary with time then for an almost circular motion we have

$$|\boldsymbol{L}| = m|\boldsymbol{r_k}||\boldsymbol{V_k}| \sin\theta \quad (21)$$

where θ is close to $\frac{\pi}{2}$
If one were to consider the angular momentum of each of the stars in the system, then the sum of the angular momenta will never change. If at t_j, star p_i has mass m at position $\boldsymbol{r}_{i,j}$ with velocity $\boldsymbol{v}_{i,j}$ then the star's angular momentum $\boldsymbol{L}_{i,j}$ is defined by

$$\boldsymbol{L}_{i,j} = m_i\left(\boldsymbol{r}_{i,j} \times \boldsymbol{v}_{i,j}\right) \quad (22)$$

In the system of N stars, let the system's angular momentum $\boldsymbol{L}_j$ at t_j be defined by

$$\boldsymbol{L}_j = \sum_{i=1}^{N} \boldsymbol{L}_{i,j} \quad (23)$$

In order to show conservation, we wish to show that

$$\boldsymbol{L}_j = \boldsymbol{L}_0 \qquad , j = 1,2,3 \quad (24)$$

From the law of vector cross product we have

$$\boldsymbol{L}_{i,k+1} - \boldsymbol{L}_{i,k} = m_i\left(\boldsymbol{r}_{i,k+1} \times \boldsymbol{v}_{i,k+1}\right) - m_i\left(\boldsymbol{r}_{i,k} \times \boldsymbol{v}_{i,k}\right)$$
$$= \Delta t\left[\frac{\boldsymbol{r}_{i,k+1} - \boldsymbol{r}_{i,k}}{2}\right] \times \boldsymbol{F}_{i,k} \quad (25)$$

6.0 Methodology

Computer simulation always involves a compromise between the number of sub systems required to enable an accurate description to be made of the essential physics, and the size and speed of the available computer on which the resulting program is to be run.
The program wants to investigate the stability or suitability of the leapfrog process in simulating the conservation of energy and angular momentum in binary stars (after the changes made to the program ''cluster'', the program was renamed ''binary'') in comparison with what was obtained in the work done with the program ''cluster'' by Woolfson and Pert [2].
The simulation stores the initial positions and velocities in a data file; START.DAT it then goes ahead to determine the geometric moment of inertia, with the position and speed of the centre of mass and the kinetic energy. It takes an initial predictor-corrector step giving only the coordinates at time Δt. With this it calculates the accelerations at initial and estimated final positions and then estimates the new positions from average velocities in the time step. The leapfrog process can now be started in order to compute the new values of velocities and position. A test is carried out for time step suitability with R_{MIN2} and V_{MAX2} since they have come from different time step, the potential energy is then calculated so also the root-mean speed from the Virial theorem and stores the final positions and coordinates with the total energy in the data file FINISH.DAT
Every star is repositioned with same speed but in a random direction, and the centre of mass is put at the origin to give zero momentum and the whole simulation is started again.

7.0 Comparison of outputs from cluster and binary programmes

The 'cluster' program was run with same input values to verify the conclusions reached in the previous work done by Woolfson and Pert. With an input of $N > 100$ or $N < 2$ the program terminates and stops running, which is ideal.
With input of N=100, R ≥ 2000au and time=10,000 years the program is terminated at 5,567 years.
With input of N=100 and R ≥ 2000au, time=100,000 years the program runs slowly.
The total energy of the system increased in an irregular manner rather than remain constant when a large years range was considered as input say 100,000 years to 300,000 years with the input radius R=40,000au as shown in Figure 1.

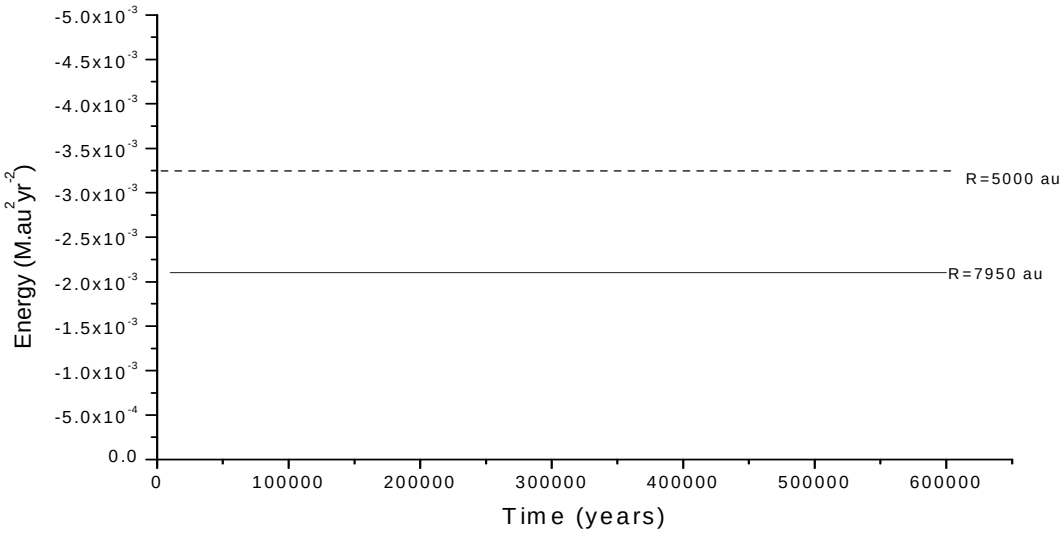

Figure 1: Plot of total energy against time for N=40, R=40000au and Time=10000 to 320000 years with input time approximations during output

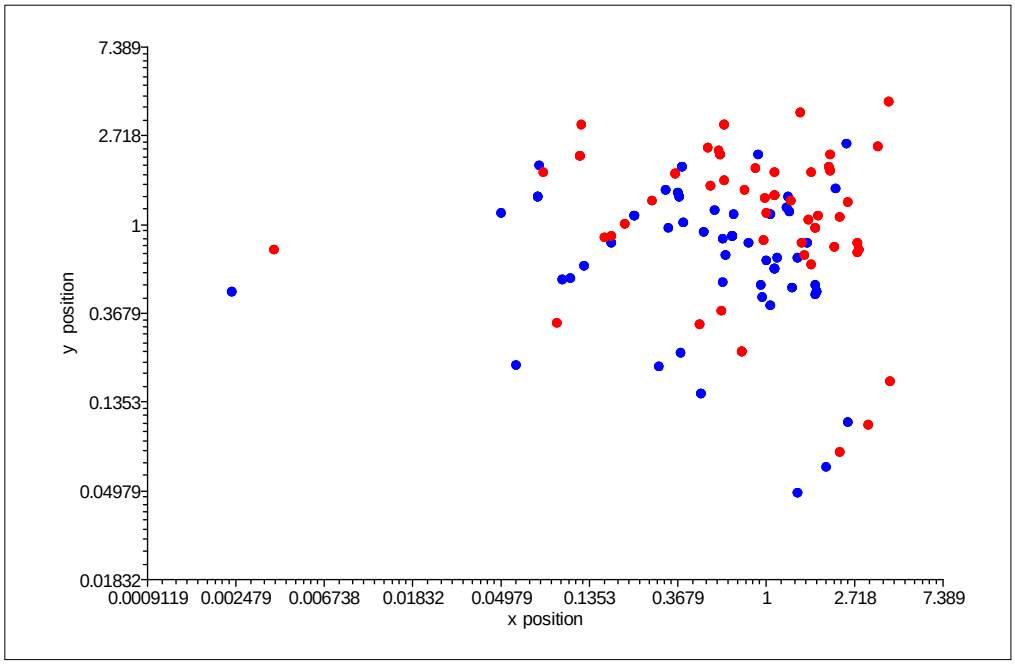

Figure 2: Plot of the 100 iterations of the cluster system for N=40, R=40000au and Time=10000 to 320000 years with input time approximations during output.

It can be seen that approximations in the years of the simulation were given as output which was possibly due to the working system of the computer used to run the program.

It will still be seen that from the total energy column of Table 3.1, something which was not necessarily conserved by the algorithm is energy, and the total energy does change and by almost 5% for the longest simulation time which verifies the work already done on ''cluster'' by Woolfson and Pert.

The program ''*binary*'' was run with inputs for number of stars N=2, and R ranging from 7950au to 500au for various simulation time ranging from approximately 0 years to 600,000 years. Figure3 and Figure 4 show the output data comprising of the total energy of the binary system, a time range of 0 years to 600,000 years and radius R within the range of 7950au to 500au.

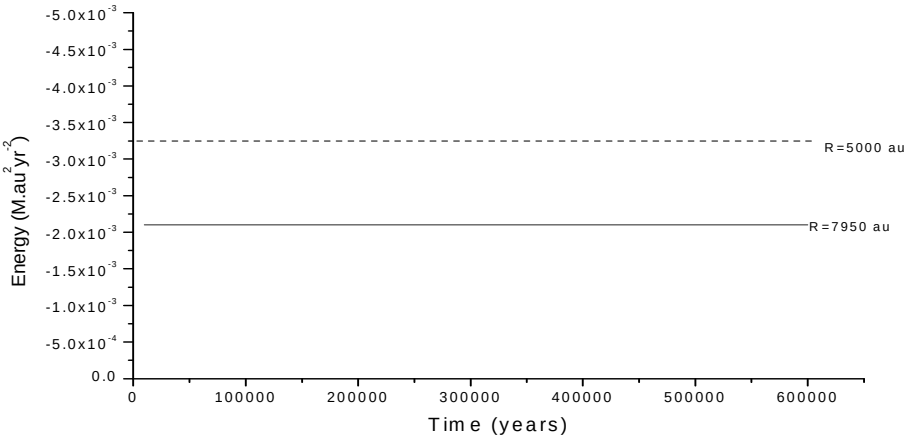

Figure 3: Plot of total energy against time for N=2, R=7950au to 5000au and time=00000 to 600000 years.

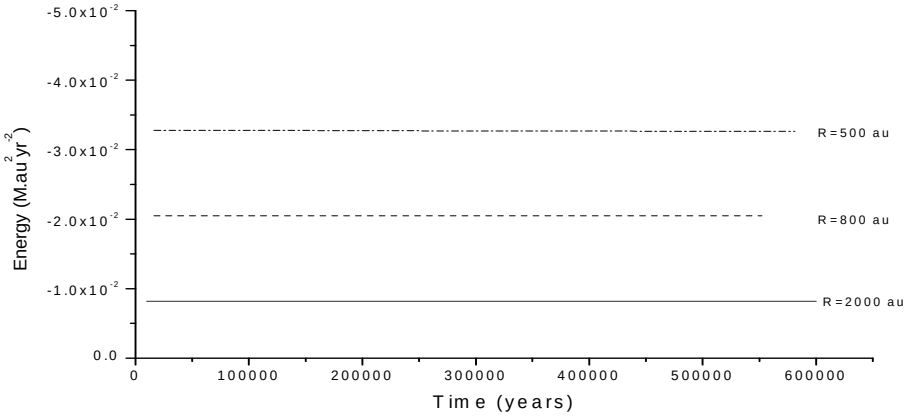

Figure 4: Plot of total energy against time for N=2, R=2000au to 500au and time=00000 to 600000 years.

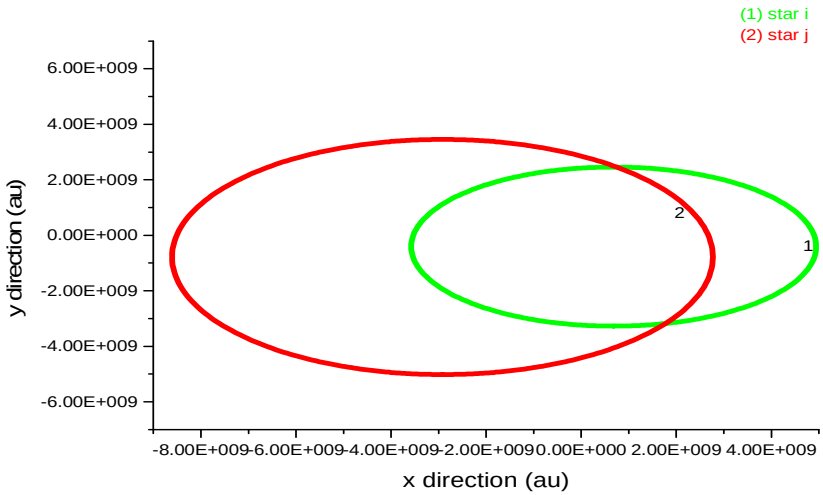

Figure 5: Plot of the elliptical orbits of the binary system relative to their x and y directions in (au)during the simulation process with N=2, R=7950au to 5000au and Time=00000 to 600000 years.

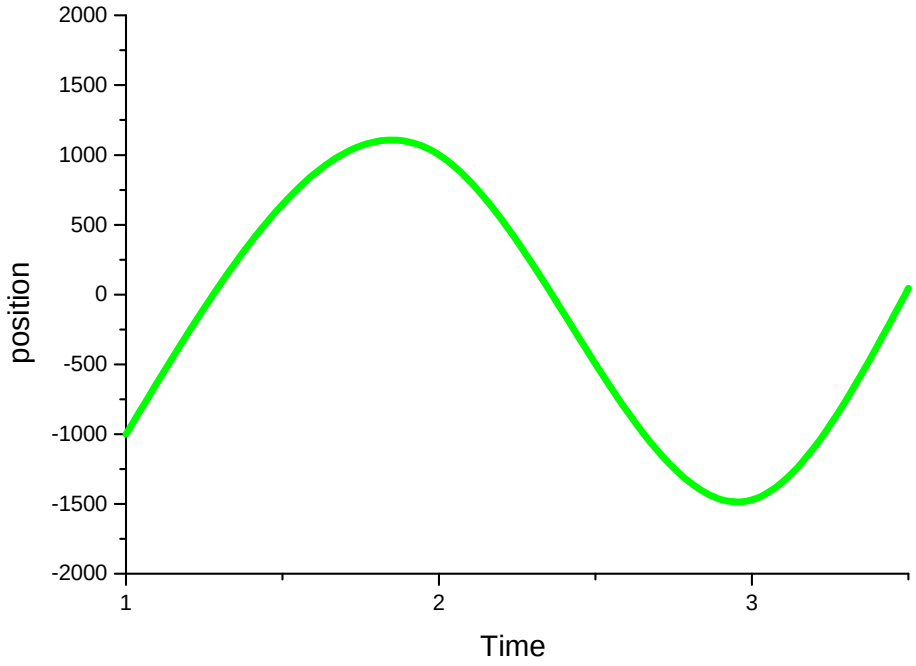

Figure 6: Plot of the position-time graph for the binary system showing the oscillatory motion of the stars during the simulated time range for Time=10000 to 320000 years with input time approximations during output, with N=40 and R=40000au.

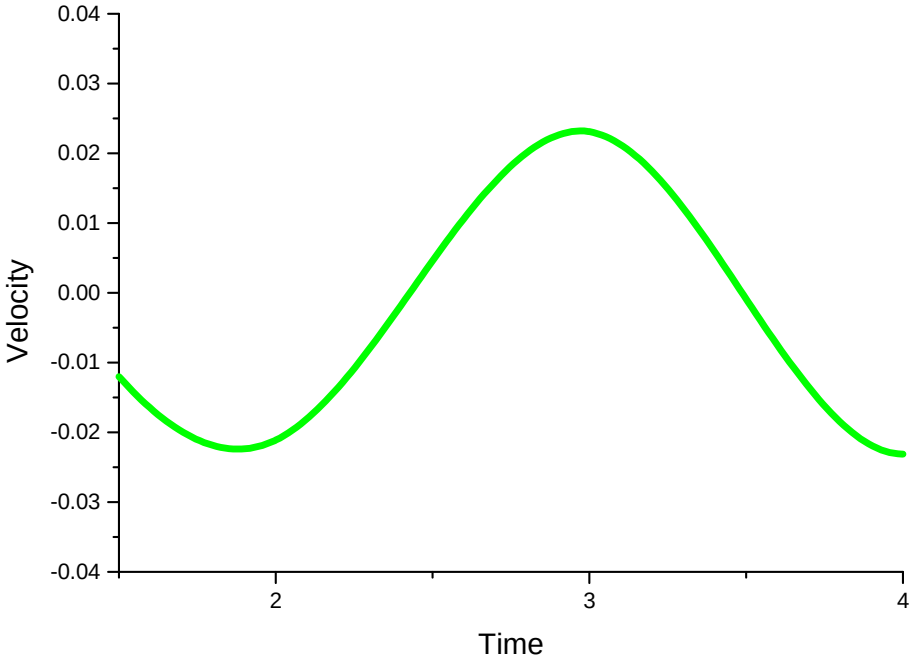

Figure 7: Plot of the velocity-time graph for the binary system showing the oscillatory motion of the stars during the simulated time range for Time=10000 to 320000 years with input time approximations during output, with N=40 and R=40000au.

8.0 Validation

The validation problem used here, deals with using the program ‘’cluster’’ to simulate a model of the solar system, which is an N-body problem comprising the motion of the Sun, the nine planets, the four moons of Jupiter and earth’s moon. The simulated orbits of these
moving bodies are comparable with NASA’s published data and figures.

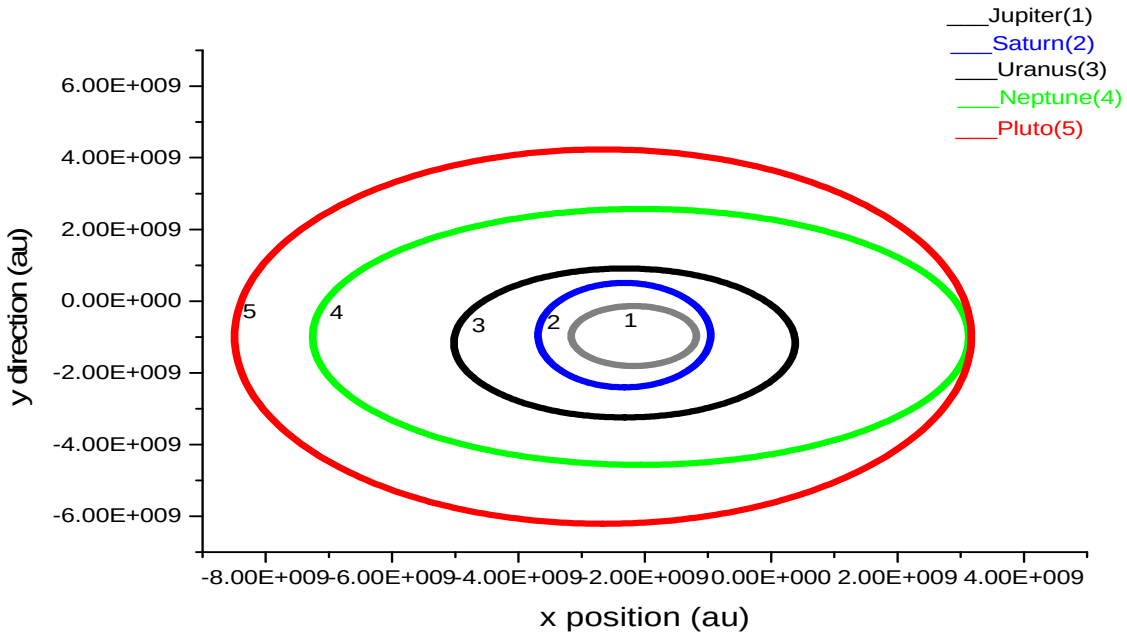

Figure 8: Plot of the simulated 15-body validation problem of the solar system showing the outer planet orbits: Jupiter, Saturn, Uranus, Neptune and Pluto, during the simulated time range for Time=10000 to 320000 years, with N=40 and R=40000au.

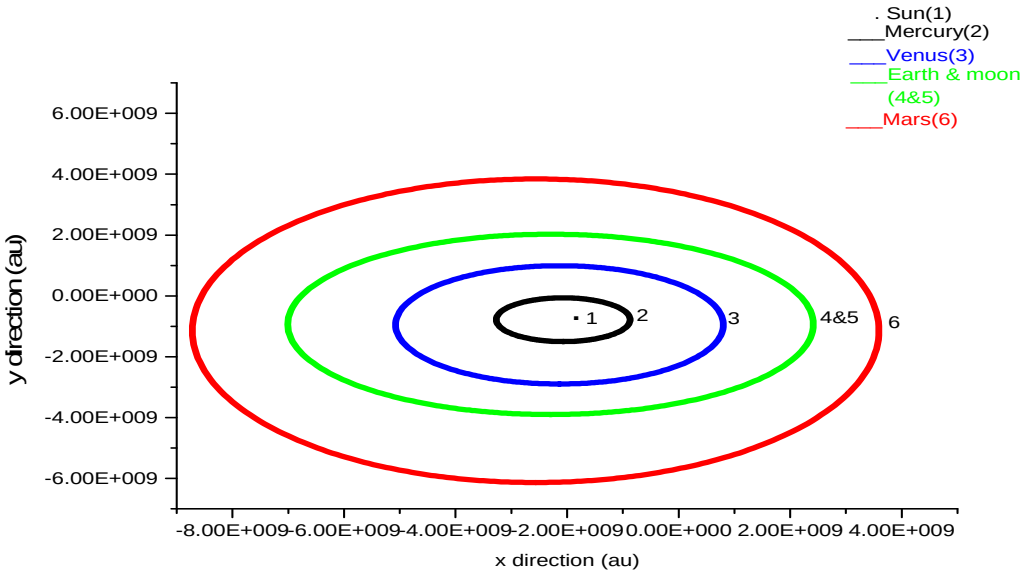

Figure 9: Plot of the simulated 15-body validation problem of the solar system showing the outer

9.0 Concluding Remarks

Models are simplified representation of real objects, physical situations or rather complex systems. This work is an example of modeling under particle simulation (few-body problem).It is assumed that if the interactions within a cluster system are symmetric, then there would be N (N-1)/2 pair interactions. The astronomical example of ''cluster'' with N=100 which was used for this work is an example of few-body system simulation.

The program used for this work incorporates two methods, the leapfrog and the Euler predictor-corrector process; the Euler predictor-corrector process starts the initial step within the simulation and proceeds to evaluate the initial boundary condition $u' = u(\Delta t)$ and thus continues with the leapfrog process which in turn is a very sensitive function of the accuracy of u'in order to minimize the errors within the simulation.

The leapfrog process shows instability in simulating energy conservation within the cluster system. Changes made to the program ''cluster'' to ''binary'' shown stability and suitability of the leapfrog process on how well it will simulate energy conservation in a system of fewer particles. It is clear that pair interactions cannot be considered to be accurate in systems of such size as the cluster or systems such as those of 10^{11} stars because the simulation of such systems will involve approximations of higher order which, may be tackled using algorithms such as the Runge-kutta method. However, in the leapfrog equation it is seen that the changes in position depend on velocity and the changes of velocity depend upon position, which is what enables the leapfrog process to be used. While this will be much less accurate than the higher order methods, it is much faster and also requires much less computer memory. If the interest is in the general behaviour pattern of the system rather than in determining the right position of each body, then the leapfrog process may be considered adequate.

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