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Mass, Density and Radius Relation in Carbon and Iron White Dwarf Stars

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Abstract

The mass- radius and mass- density relation of carbon and iron white dwarf stars was studied using the fortran code program with given stars central density. Data generated were used to plot the mass-radius and mass-density relations. The result obtained were compared with theoritical models of different star composition.

Keywords:

1.0 Introduction

White dwarf star is the end stage of the life of a low to medium mass star, such as our sun.These unusual stars begin as average, sun-like stars, slowly burning hydrogen in the core. By the time a star like this has reached the end of its lifetime, it has exhausted its store of hydrogen and helium and is left with a core of carbon. This carbon remains inert due to the large barrier required for fusion and sits at the centre of the core. If the star cannot adjust its temperature to over come the fusion barrier, the surface layers will be expelled and the core will remain behind, isolated in space. This remnant is known as a carbon white dwarf.[1- 3, 5 - 10]

Thermonuclear fusion drives star through many stages of combustion, the hot centre of the star allows hydrogen to fuse into helium. Once the core burned all available hydrgen, it will contract until another source of support becomes available. As the core contracts and heats, transforming gravitational energy into kinetic or thermal energy, the burning of the helium ashes begins. For stars to burn heavier elements, higher temperatures are necessary to overcome the increasing Coulomb repulsion and allow fusion through quantum-mechanical tunneling.Thermonuclear burning continues until the formation of an iron core. Accordingly , the composition of white dwarf stars is mostly dominated by its nuclear ashes.in the standard scenario white dwarfs are composed mostly of carbon and oxygen, but there is observational evidence that helium and iron white dwarfs exixt as well. As there is no fusion processes taking place in the interior of the white dwarf star, a different force than thermal pressure are needed to keep the star in hydrostatic equilibrium. Due to Pauli Exclusion Principle, two fermions cannot occupy the same quantum state. With increasing density, the electrons fill up the phase space from the lowest energy state i.e. the smallest momentum. Consequently, the remaining electrons sit in physical states with increasing momentum. The resulting large velocity leads to an adequate pressure of the electrons, the degeneracy pressure counterbalancing gravity's pull and stabilizing the white dwarf. [2].

The pressure throughout the star is due to degenerate electrons , which are nonrelativistic in the outer layers, but are relativistic in the intrior of the more massive stars with $M \sim M_0$.

The degeneracy of electrons plays a crucial role in accounting for the properties of white dwarfs and causes these stars to be markedly different in many ways from ordinary gaseous stars. It is interesting to note that the white dwarf stars exhibit, on a macroscopic scale and in a very striking way, both a quantum-mechanical effect and a special relativity effect.

White dwarfs are expected to have a radii of the order of 10km-roughly the size of the Earth, or about one percent of that of the sun(which radius $R_0 \approx 7 \times 10$^5 km. The small radii and relatively large masses imply enormous mean densities of order of 10^8 kg m^{-3} to 10^{11} kg m^{-3} (a thimble full of this material on earth surface might weigh as much as locomotive) [3].

The purpose of this study is to present the mass-radius and density - radius relations of carbon and iron white dwarf stars at a given central densities using code program developed by [4].

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2.0 The Equilibrium Equation

Assuming cold, spherically symmetric and non-rotating white dwarfs in hydrostatic equilibrium, Weinberg [5] used simple hydrostatics to get the equations of stellar equilibrium:

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2}, \quad P(r=0) \equiv P_c \;, \tag{2.1}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho(r), \qquad m(r=0) \equiv 0, \tag{2.2}$$

where

G = gravitational constant

$\rho(r)$ = density of the star at r

$m(r)$ = mass contained with r

P=pressure

Thus, given an equation of state $P = P(\rho)$, we can obtain the mass-radius relationship of the given of the given object by integrating up to the surface indicated by P=0.

2.1 Gravity and Pressure.

For a cube of matter in the white dwarf, of volume δV, the force, F on the cube is

$$F = -\frac{Gm(r)}{r^2}\rho(r)\delta V, \tag{2.3}$$

The force per unit volume is due to the change in pressure with radius:

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2}. \tag{2.4}$$

We can write density as a function of radius using:

$$\frac{dP}{dr} = \left(\frac{d\rho}{dr}\right)\left(\frac{dP}{d\rho}\right) \tag{2.5}$$

Equating Eq. (4) and Eq. (5), we have

$$\frac{d\rho}{dr} = -\left(\frac{dP}{d\rho}\right)^{-1}\frac{Gm\rho}{r^2} \;, \tag{2.6}$$

also, we can write the mass as a function of radius:

$$\frac{dm}{dr} = 4\pi r^2 \rho(r). \tag{2.7}$$

Eq. (2.6) and (2.7) are two coupled first order differential equations. The boundary conditions at r are:

$$\rho = \rho\,(r=0) = \rho_c \tag{2.8}$$

$$m(r=0) = 0. \tag{2.9}$$

2.2 Equation of State.

In order to solve the differential equations(6) and (7), we must find pressure as a function of density, also known as the equation of state of the white dwarf. When the temperature of the star is low enough, the electrons fill the lowest possible quantum levels. Since the nuclei are very heavy compared to the electrons, we can assume that most of the mass of the star comes from the nuclei. The electrons are moving much faster than the nuclei, so most of the pressure comes from the electrons. The density in the star is so high that the electrons are not bound to individual nuclei, but rather move freely about the star. Therefore, a good model for the white dwarf star is a free Fermi gas of relativistic electrons at zero temperature.

The number density of electrons, n_e is given by

$$n_e = \frac{N}{V} = Y_e \frac{\rho}{m_N}, \tag{2.10}$$

where, N=number of electrons, V=volume occupied by the electrons Ye = electrons fraction and m_N = nucleon mass.

The nucleon mass from now on will be treated as the mass of a proton, since protons and neutrons have nearly the same mass. [6].

The mass density of dense matter given by Eq. (2.8) can be expressed in terms of the electron number density as:

$$\rho = \frac{m_p n_e}{Y_e}. \tag{2.11}$$

For a pure substance, it is just Y_e = Z/A, but for a mixed substance, where a variety of nuclides are present, it is given by

$$Y_e = \sum_i X_i \left(\frac{Z_i}{A_i} \right)_i, \tag{2.12}$$

where, the subscript i designates the nuclide type and x_i denotes the relative abundance of that nuclide in the system. For example, Y_e = 1.0 for a pure hydrogen system, Y_e = 0.5 for a pure ^{4}He system, and Y_e = 26/56 = 0.464 for a pure ^{56}Fe system. All pure substances that are composed of nuclides lying between He and Fe in the atomic periodic table have electron fractions bounded by the values 0.464 and 0.5. Since pressure depends on n_e of a system while its mass density depends on n_e/Y_e, it is clear that matter systems with the same Y_e would have the same equation of state. [7].

2.3 Fermi Momentum.

Let us derive an expression for the Fermi momentum in terms of the variables stated above. There are two spin states per level, and levels per unit volume

$$p = \frac{4\pi p^2 dp}{h^3}, \tag{2.13}$$

The number of levels per unit volume is the density of states. Integrating Eq. (2.13) to the Fermi momentum, we have

$$n_e = \frac{2 \times 4\pi}{h^3} \int_0^{p_f} p^2 dp = \frac{8\pi}{3h^3} p_f^{\ 3}. \tag{2.14}$$

Equating Eq. (2.10) and Eq. (2.14),

$$p_f = h \left(\frac{3 Y_e \rho}{8\pi m_p} \right)^{\frac{1}{3}}. \tag{2.15}$$

2.4 Relativistic Kinetic Energy of the Electrons.

The relativistic kinetic energy and the relativistic speed of the electrons is given by

$$\varepsilon = m_e c^2 \left[\left(1 + \left\{ \frac{p^2}{m_e^{\ 2} c^2} \right\}^{\frac{1}{2}} \right) - 1 \right], \tag{2.16}$$

$$u = \frac{d\varepsilon}{dp} = \frac{\left(\frac{p}{m_e} \right)}{\left(1 + \left\{ \frac{p^2}{m_e^2 c^2} \right\} \right)^{\frac{1}{2}}}. \tag{2.17}$$

where, c is the speed of light, m_e is the mass of electrons and p is the momentum.[6].

2.5 The Pressure of the Electrons.

The pressure the electron supplied is given as,

$$P = \frac{1}{3} \frac{N}{V} \langle pu \rangle = \frac{8\pi}{3h^3} \int_0^{p_f} \frac{\left(\frac{p^2}{m_e} \right)}{\left(1 + \left\{ \frac{p}{m_e c} \right\}^2 \right)^{\frac{1}{2}}} p^2 dp. \tag{2.18}$$

By introducing a new variable

$p = m_e\, c \sinh(\theta)$ (2.19)

$u = c \tanh\ (\theta)$ (2.20)

and substitute these into Eq. (2.12) gives

$$n_e = \frac{8\pi m_e{}^3 c^3}{3h^3} x^3, \tag{2.21}$$

$$x = \sinh\ (\theta_f) = \frac{p^f}{m_e\ c} = \left(\frac{3n}{8\pi}\right)^{\frac{1}{3}} \left(\frac{h}{m_e c}\right). \tag{2.22}$$

So, pressure can be written as

$$P = \frac{8\pi m_e{}^4 c^5}{3h^3} \int_0^{\theta_f} \sinh(\theta)^4 d\theta = \frac{\pi m^4{}_e c^5}{3h^3} A(x), \tag{2.23}$$

where

$$A(x) = x(x^2+1)^{\frac{1}{2}}(2x^2-3) + 3\sinh^{-1}(x). \tag{2/24}$$

To find x for our model, we note that $x = \frac{p_f}{m_e c} = \left(\frac{n}{n_o}\right)^{\frac{1}{3}} = \left(\frac{\rho}{\rho_o}\right)^{\frac{1}{3}}$, (2.25)

where $n_o = \frac{8\pi}{3} \frac{m_e{}^3 c^3}{h^3}$, (2.26)

and

$$\rho_o = \frac{m_p n_o}{Y_e}\ . \tag{2.27}$$

Now, the equation for pressure is:

$$P = \frac{Y_e m_e c^2}{8m_p} A(x), \tag{2.28}$$

and the equation of state is:

$$\frac{dP}{d\rho} = \frac{Y_e c^2 m_e}{m_p} \gamma(x), \tag{2.29}$$

where

$$\gamma(x) = \frac{x^2}{3(1+x^2)^{\frac{1}{2}}}. \tag{2.30}$$

The final differential equations are:

Mass: $\frac{dm}{dr} = 4\pi r^2 \rho(r).$ (2.31)

Density: $\dfrac{d\rho}{dr} = -\dfrac{Gm(r)\rho(r)}{r^2}\dfrac{m_p}{m_e}\dfrac{1}{Y_e c^2}\dfrac{1}{\gamma(x)}.$ (2.32)

These can be solved numerically for the structure of white dwarf stars, given the boundary conditions, for different values of the central density

2.6 Scaling the Equation.

To appropriate scale the equation (2.31 and 2.32) of white dwarf star structure, we introduce dimensionless radius, density, and mass variables as,

$$r = R_o\bar{r}, \qquad \rho = \rho_o\bar{\rho}, \qquad m = M_o\bar{m}, \tag{2.33}$$

Substituting into Eq. (2.29, and 2.30) and using Eq. (2.28) yields, after some arrangement,

$$\frac{dm}{dr} = \left(\frac{4\pi R_o{}^3\rho_o}{M_o}\right)\bar{r}^2\bar{\rho}\,; \tag{2.34}$$

$$\frac{d\bar{\rho}}{dr} = -\left(\frac{GM_o}{R_oY_e\left(\dfrac{m_e}{M_p}\right)}\right)\frac{\bar{m}\bar{\rho}}{\gamma\bar{r}^2}. \tag{2.35}$$

If we now choose M_o and R_o so that the coefficients in the parentheses of these two equations are unity, we find that

$$R_o = \left[\frac{Y_e(\dfrac{m_e}{M_p})}{4\pi G\rho_o}\right]^{\frac{1}{2}} = 7.72\times10^8 Y_e \text{ cm}, \tag{2.36}$$

$$M_o = 4\pi R^3{}_o\rho_o = 5.67\times10^{33}Y_e \text{ gm}, \tag{2.37}$$

and the dimensionless differential equations are

$$\frac{d\bar{p}}{d\bar{r}} = -\frac{\bar{m}\bar{\rho}}{\gamma\bar{r}^2}; \tag{2.38}$$

$$\frac{d\bar{m}}{d\bar{r}} = \bar{r}^2\bar{\rho}. \tag{2.39}$$

By defining the natural mass and length scales of the system (M_o and R_o) according to Eq. (2.36) and (2.37) the two long expressions in brackets in the above equation are taken as unity. Finally, the equations of hydrostatic equilibrium describing the structure of white dwarf stars are given by Eq. (2.38) and 2.39).

These coupled set of first order differential equations are solved using standard numerical techniques, such as the Runge-Kutta algorithm.[4]

2.7 The Chandrasekhar Limit.

By using the polytrophic equation $P(\rho) = K\ \rho^{\Gamma}$),[8] was able to derive some basic features of white dwarf stars:

$$\rho_c \le 10^9\text{kgm}^{-3}\ \ \Gamma = \frac{5}{3} : \text{R} \cong 1.12\times10^6\times(\frac{\rho_c}{10^9}\text{kgm}^{-3})^{-1/6}\times(\frac{Y_e}{0.5})^{5/6}\text{m} \tag{2.40}$$

$$M = 0.496\times(\frac{\rho_c}{10^9}\text{kgm}^{-3})^{1/2}\text{x}(\frac{Y_e}{0.5})^{5/2}M_o \tag{2.41}$$

$$\rho_c \ge 10^9\text{ kgm}^{-3}\quad \Gamma = \frac{4}{3} : \ \text{R} \cong 3.35\times10^6\times(\frac{\rho_c}{10^9}\text{kgm}^{-3})^{1/3}\times(\frac{Y_e}{0.5})^{2/3}\text{m} \tag{2.42}$$

$$M \cong 1.46 \times (\frac{Y_e}{0.5})^2 M_o \qquad (2.43)$$

Equation (2.43) shows that there is no dependence of M on the central densities. The mass $M_{ch} \cong 1.46M_o$ is the Chandrasekhar mass i.e. is the maximum mass a white dwarf can reach if it achieves sufficiently high densities. So that it's entire structure is governed by relativistic fermions. In practice this cannot be reached since the outer layer have non relativistic electrons [9]. Hydrostatic equilibrium as a cold body becomes impossible to achieve when $M>M_{ch}$, as the core will collapse and supernovae shock develops ejecting most of the mass of the star into interstellar space and leaving behind an extremely dense core – the neutron star. Beyond this limiting mass no source of pressure exists that can prevent gravitational contraction. If such is the case, then the star will continue to collapse into an object of zero radius: a black hole. [10].

3.0 The Code

A FORTRAN program developed by [4] was the main program used for all computations. It constructs a series white dwarf star models for a given electron fraction, Y_e with central densities ranging in equal logarithimic steps between the values of DEN1 and DEN2 specified. For a chosen star model, the dimentionless equation, Eq. (38 and 39), with are the first order differential equation for mass and density are intergrated using the fouth order Runge-Kutta Algorithm. An empirically scaled radial step DR is used for each model, and initial conditions for each integration are determined by Taylor expansion of the differential equation about r=0 (beginning of the same DO loop). Integration proceeds until the density has fallen below 10^3 gm cm^{-3}. . In addition, the contributions to the internal and gravitational energies are calculated at each radial step.

3.1 Input

The inputs parameters include the electron fraction, the central density for the first model, the central density for the last model, and the number of models to be calculated.

3.2 Output

As each model is calculated, the radial step, central density, number of steps, total radius, total mass, kinetic energy, gravitational energy, and total energy for the model are displayed. Densities are in grams per cubic centimeter, energy is in joules, distances are in centimeters, and mass in grams. After all the models are integrated, the graphics display two plots. The first is the total mass of the model vs. the central density; the second is the density vs. radius for each model.

4.0 Result and Discussion

Data and graphs were generated and plotted for carbon, iron and theoretical models of white dwarf stars with Y_e value of 0.5, 0.464 and 0.6, 0.7 0.8 respectively. The mass –radius and density-radius relationship were plotted using graphic software *origin 5.0*. The graphical result obtained in Fig 1a were studied and compared with Fig 1b scanned mass-radius relation of carbon and iron white dwarf stars obtained using Excel and maple numerical technique

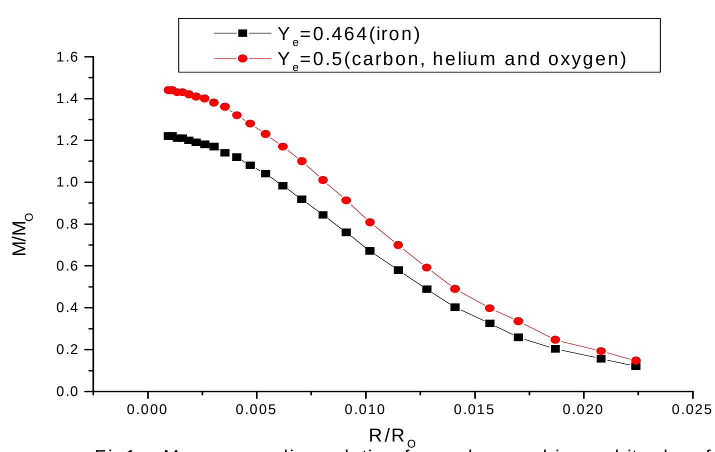

Fig1a: Mass vs. radius relation for carbon and iron white dwarf stars for central density of 10^{11}gm/cm^3.

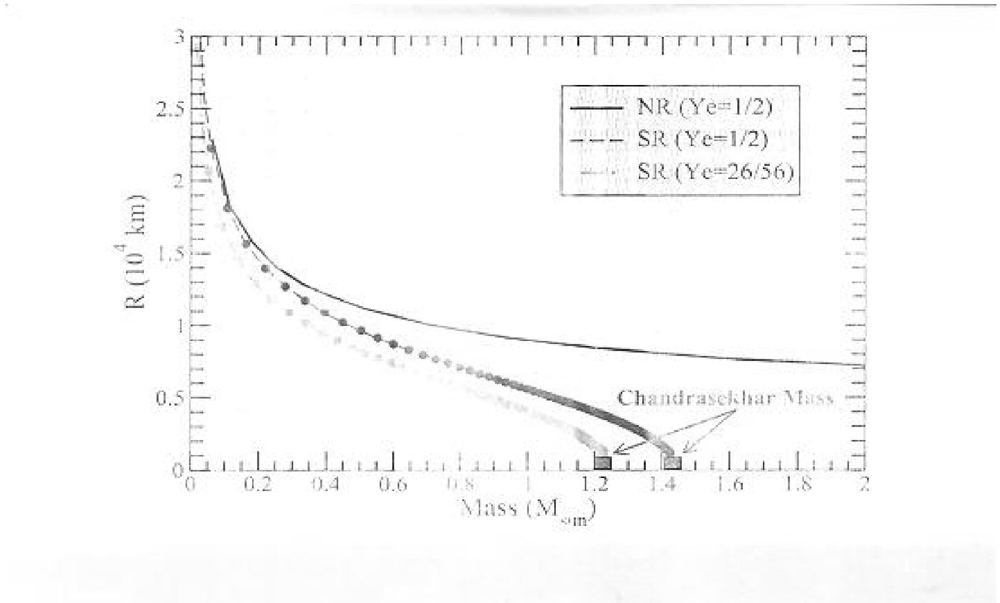

Fig 1b: Mass-*vs*-radius relation for white dwarf stars obtained using Excel (lines) and Maple(symbols), scan from [10].

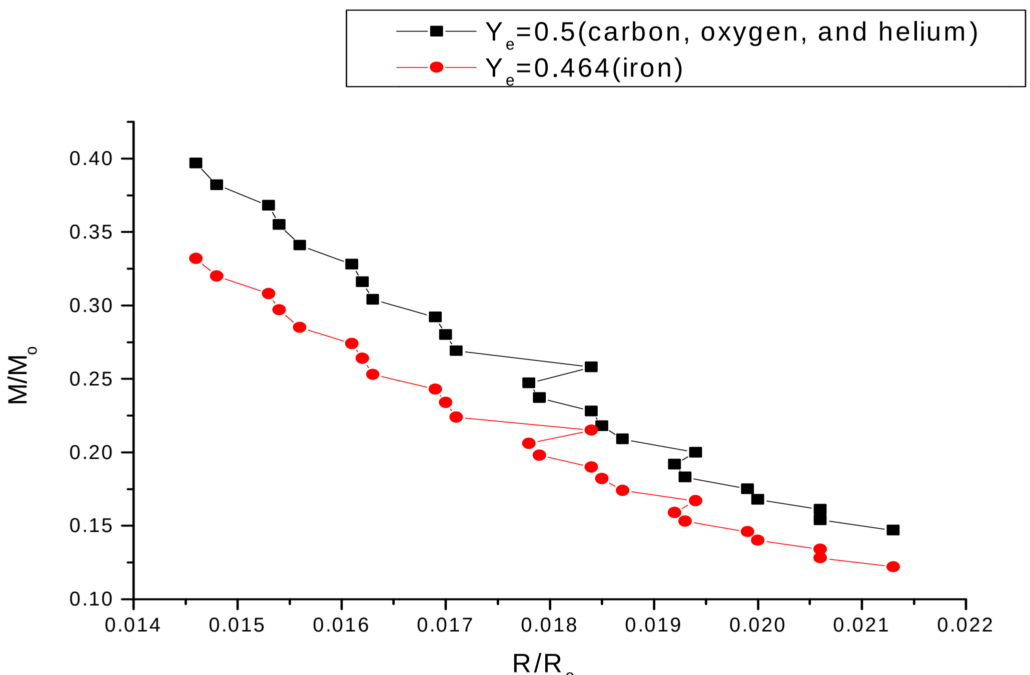

Fig 2:Mass vs. radius relation of carbon and iron white dwarf stars for central densities of 10^6g/cm (non-relativistics).

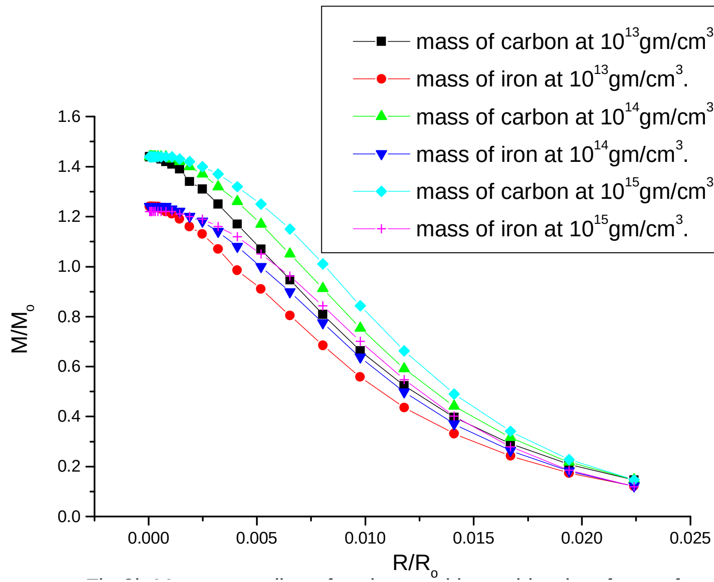

Fig 3: Mass vs. radius relation of carbon and iron white dwarf stars for central densities of 10^{12}, 10^{14} and 10^{15} g/cm^3.

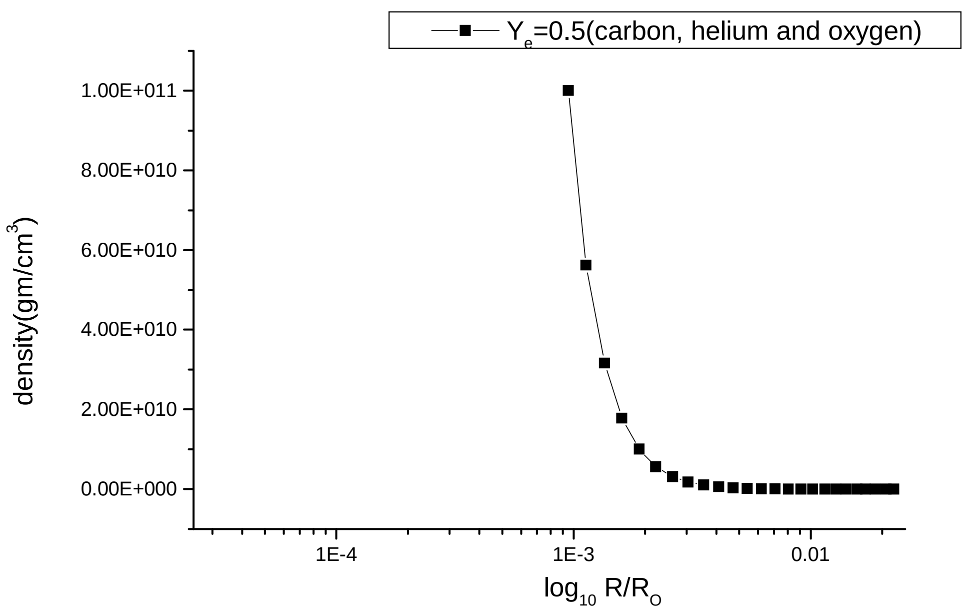

Fig4: Density vs. radius for carbon, helium and oxygen white dwarf stars.

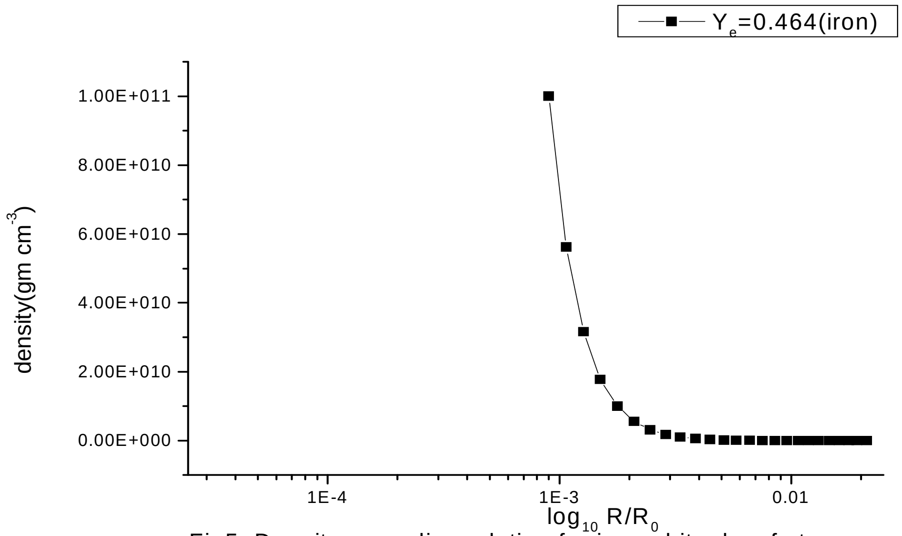

Fig5: Density vs. radius relation for iron white dwarf star.

4.1 The Mass –Radius Relationship.

The mass - radius relation for carbon, helium, oxygen and iron Fig1a for electron fraction of Y_e=0.5 and 0.464 respectively, obtained in this work agrees with the one obtained by [10], using excel and maples codes Fig1b. In both cases the same maximum mass limit of $1.44M_o$, and $1.2M_o$ for carbon and iron white dwarf stars respectively were obtained. From Fig 1a it shows that as the central density of a white dwarf star increases its mass approaches a limiting value M_{ch} and the star becomes very small. M_o= solar mass = 1.98 x10^{30} kg and R_o= solar radius = 6.95 x10^8 m.

The Maximum mass limit is not achieved when the speed of the electrons is non- relativistic i.e. $\rho_c \leq 10^9$ kgm^{-3} as shown in Fig 2a. By increasing the central densities to many degrees of magnitude i.e. ρ_c =10^{16}, 10^{17}, and 10^{18} kgm^{-3}, Fig 2b, the mass remains in relatively narrow region. Chandrasekhar limit is still obeyed, eventually reaching but not exceeding the Chandrasekhar limit. Fig 3 shows the maximum mass limits of $2.03M_{o,}$ $2.82M_{o,}$ and$3.69M_o$ for theoretical models of white dwarf stars with Y_e= 0.6, 0.7, and 0.8 respectively. These maximum mass limits of $2.03M_{o,}$ $2.82M_{o,}$ and$3.69M_o$ also agree with the Chandrasekhar mass obtained in Eq. (2.43).

4.2 Density-Radius Relationship.

The density -radius relation in Fig 4 and 5 for carbon and iron white dwarf stars for central density range of 10^{14} kgm^{-3}, shows that as the central density increases the mass also increases while the radius decreases and the star becomes more centrally concentrated. The minimum radius obtained for carbon and iron white dwarf was 9.51x10^{-4} solar radius and 8.9x10^{-4} solar radius respectively.

5.0 Conclusion

From the graphical results obtained, the maximum mass of carbon and iron white dwarf stars agrees with the one obtained by Chandrasekhar $M \cong 1.46 \times (\frac{Y_e}{0.5})^2 M_o$, (Eq. 43). The program works and can be used to find the maximum mass, radius for any white dwarf stars of known Y_e at a given central density. Prediction regarding the properties of white dwarf stars serves to test our understanding of matter at high densities. In this work we have described the tight interconnection between the macroscopic and microscopic properties of white dwarf stars at high densities, while the fundamental principles of cold high density matter are believed to be understood, key question remains and new observations may give rise to new puzzles.

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