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Treatment of Longitudinal Waves in Plasma Using Poison's Equation

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Abstract

Wave phenomenon in plasma is different from what is obtained in ordinary air. The nature of plasma and the natural oscillations in plasma make wave propagation in it more important, for example, wave interactions with charged particles have been of great importance in the development of controlled thermonuclear reactions where high frequency electromagnetic fields are used in accelerating higher particles for nuclear bombardments to activate nuclear reactions. General treatments of wave particles interaction involving magneto-hydrodynamic equations of charged particles in arbitrary wave fields are mathematically very involving and cumbersome. This study considered propagation of waves in plasma and the simplest way of determining its parameters such as speed, frequency and wave number using Poison's equation instead of the cumbersome full Maxwell's equations.

1.0 Introduction

The term "Plasma" first appeared in physics that is, a gas which contains a noticeable proportion of charged particles (electrons and ions). To understand the conditions of plasma formation, it is very necessary to compare plasma and mixture of chemically active gases. For instance, the following chemical reaction can occur in the air, which is basically a mixture of nitrogen and oxygen. [1]

$$N_2 + O_2 \Leftrightarrow 2NO - 41.5\,Kcal/mole \qquad (1)$$

Hence, a small amount of nitric oxide,(NO) is present in the air at the equilibrium between nitrogen and oxygen. According to the Le'chateiers' principle, increasing the air temperature result. In a larger equilibrium amount of nitric oxide.

The equilibrium between the neutral and charged particles is similar to the above case, an atom or molecule consists of bound positively charged nuclei and negatively charged electrons. Thus, plasma is an electrically neutral collection of electrons and positive ions [2] .

Plasma Dynamics

Plasma dynamics is the study of the dynamics of ionized gases, especially of fully ionized gases. The most familiar terrestrial ionized gases occur in electronics arcs in which usually are a modest degree of ionization is present, so that the ionized component can be considered to diffuse off through the neutral gas [2].In the absence of a magnetic field, the gross dynamic behaviour of fully ionized plasma differs form that of a normal gas only in its electrical properties. At low frequencies the plasma acts as an electrical conductor and does not acquire large electric charges, any accumulations being neutralized by an electron flow in terms of order ω_p

Where $\omega_p = \sqrt{\dfrac{4\pi^2}{m}} = 10^5 \dfrac{me}{s}$ (2)

For frequencies > ω_p

The electrical behaviour of the plasma is summarized by the dielectric coefficient

$$E = 1 - \omega_p{}^2 \frac{2}{\omega^2} \qquad (3)$$

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An attempt to develop a detailed kinetic theory of the plasma encounters a serious difficulty through the long range nature of the coulomb force which makes a representation of particle interaction by two particle encounters unrealistic, the Boltzmann collision integrals diverging. Those correlations are properly treated, since the principle effect of correlations is its screen, the interaction between distance, charges and a simple Boltzmann equation screened at the Debye length.

$$\lambda_0 = \left[\frac{KT}{m\omega^2 p}\right]^{\frac{1}{2}} \qquad (4)$$

Plasma dynamic becomes much more involved when a magnetic field is present described as magnetohydodynamics [3].

Plasma Frequency

The resultant space charge in the plasma between the cathode dark space and the anode of an electric discharge tube, the concentration of electrons and positive ions is high and approximately equal, here, is nearly zero.

If the neutrality of charge is disturbed, the electrons will oscillate about their equilibrium position as they are much higher than the positive ions. Considering the positive ions, to have fixed axis positively, the electron execute simple harmonic motion with the plasma frequency given by [4] $\omega_p{}^2 = \frac{ne}{m\varepsilon_o}$ (5)

Where 'n' is the electron density in the electron mass, ε_o is the permittivity in free space.

Origin of Waves in Plasma

Originally, Maxwell's equations lead to propagation of electromagnetic energy in

i. Vacuum
ii. Dielectric medium
iii. Conducting medium

These equations are categorized into four.

The first Maxwell's equation is as a result of Faraday's law of induction expressed as $E = \frac{-\partial\phi}{\partial t}$ where E is electromotive force.

ϕ is magnetic flux

But $E = \oint E.dl.dl$ and $\partial\phi = B.dS$

B.S is magnetic induction and S is a vector.

Therefore $E = \oint E.dl = -\int \frac{\partial B}{\partial t}\partial S$

Which give curl $E = \frac{-\partial B}{\partial t}$

That is, $\nabla \times \mathrm{E} = \frac{-\partial \underline{B}}{\partial t}$

This implies that in a region where magnetic field changes with time, an electric fried is set up.Hence, this Maxwell's equation is expressed as

$$\Delta \times \mathrm{E} + \frac{1}{C}\frac{\partial B}{\partial t} = 0 \qquad (6i)$$

Where C is the speed of light in free space

The second Maxwell's equation is of force applied to electric field which is expressed as follows:

$$\int_{es} E.ds = \frac{1}{\varepsilon_0}\int \rho \partial v$$

ρ is charge density, v is volume, stokes theorem, $\int_{es} E.ds = \frac{1}{E_0}\int_v divE.dv$

$$divE = \frac{\rho}{\varepsilon_0}$$

This equation is expressed as

$$\Delta.E = 0 \qquad (6ii)$$

The third Maxwell's equation is also known as Maxwell's magnetic flux equation, it is also called a consequence of inverse square law of force applied to magnetic field. This is expressed as $\int_{cs} B.ds = 0$

This can as well be expressed as $divB = 0$ or

$$\Delta.B = 0 \qquad (6iii)$$

The forth Maxwell's equation is also called Maxwell's equation for induced magnetic field.
From the Ampere's law

$$\oint B.dl = \mu_o l$$

Where $\mu_0 =$ permeability, i =conduction current or displacement current where j is current density and it is current per unit area $(j = i/A)$

But in the case of alternate current, I_{ac} is related to the rate of charge of surface change density.

$\frac{d\sigma}{dt}$ on the surface of the capacitor.

Therefore I in equation $\oint B.dS = \mu_0 I$,is the sum of conduction and displacement current of surface charge σ. Displacement current $I = \frac{dq}{dt} = A\frac{d\sigma}{dt}$ where q is charge, σ is surface charge density or displacement current (j) $= \frac{d\sigma}{dt}$

Between the plate of parallel capacitors

$$E = \frac{\sigma}{\varepsilon_o}$$

Displacement current $(j) = \varepsilon_o \frac{\partial E}{\partial t}$

Therefore, $\oint \underline{B}.dl = \mu_0\varepsilon_0 \int \frac{\partial \underline{E}}{dt}, ds + \mu_0 \int j.d\underline{s}$

Or $\oint \underline{H}.ds = \int \frac{\partial D}{\partial t}.ds + \int ids$

Where $\underline{B} = \mu_0 H$, as B is magnetic induction or magnetic flux density which is flux per unit area $\frac{\phi}{A}$.

$\phi = \int \underline{B}.ds$, which is the flux

$\underline{B}$ is flux density and its unit is wb/m^2 or Tesla(T).

$\underline{H}$ is magnetic field intensity Am^{-1}

$\underline{D}$ is electric displacement $= \varepsilon_0 E$

Hence, $\underline{D} = \varepsilon_0 E$,where $\underline{E}$ is the electric field intensity.

The differential forms of the equation is

$$curl\underline{E} = \mu_0\varepsilon_o \frac{\partial E}{\partial t} + \mu_0 j \qquad (6iv)$$

,or

$$curlH = \frac{\partial D}{\partial t} + j$$

Hence, the Maxwell's fourth equation states that in a region of space in which the electric field is changed, a magnetic field is produced [4] and [5] .

In a more compact form, these equations (6i to 6iv) can be expressed as

$$\nabla.E = 0$$
$$\nabla.B = 0$$
$$\nabla.E + \frac{1}{C}\frac{\partial B}{\partial t}$$
$$\nabla.B - \frac{1}{C}\frac{\partial E}{\partial t} = 0$$

How Maxwell's Equations give rise to Travelling Wave in Free Space

Using

$$\nabla \times E = -\frac{\partial \underline{\underline{B}}}{\partial t} \qquad (i)$$

$$\nabla \times B = \mu_o \varepsilon_o \frac{\partial E}{\partial t} \qquad (ii)$$

Taking the curl of equation $\nabla \times \nabla \times E = \frac{-\partial}{\partial t}(\nabla \times \underline{B}).$

$$= -\mu_0 \varepsilon_0 \frac{\partial^2 \underline{\underline{E}}}{\partial t^2} \qquad (iii)$$

But $\nabla \times (\nabla \times E) = \nabla(\nabla.E) - \nabla^2 E)$ (iv)

and in the absence of charge, $\nabla.E = 0$

Hence, equation (iv) $\nabla^2 E = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t}.$ (v)

Which is the wave equation in three dimensions because ∇ is 3- dimensional operation

$$\nabla = \underline{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial Z}$$

In one dimension as for PLANE WAVE, plane wave in x- direction, Electric field 'E' does not vary with y to z

Hence, $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$

Therefore,

$$\frac{\partial^2 E_y}{\partial x^2} = \mu_o \varepsilon_o \frac{\partial^2 E_y}{\partial t^2} \qquad (vi)$$

Similarly,

$$\frac{\partial^2 B_z}{\partial x^2} = \mu_o \varepsilon_o \frac{\partial^2 B_z}{\partial t^2} \qquad (vii)$$

Waves in Plasma

Although plasma, as consequence of the interaction between particles, is gas oscillations and noise play a much greater role than in ordinary gas. First, in plasma that is located in external fields and is not homogeneous, a wide variety of oscillations between particles occur because of the long-range interaction between particles.

Secondly, these oscillations vary frequently and become amplified to a relatively high energy. In this case, the plasma oscillation determines its parameters and development.

**ACOUSTICS OSCILLATIONS.
The natural vibrations of gas are the acoustic vibrations, that is the waves of compression and rarefaction which propagates in gas. The frequency and wavelength (f and λ) respectively of vibrations are related to the wave vector K as follows

$$K = \frac{2\pi}{\lambda}$$

Any macroscopic parameter of the system can be expressed as

$$A = A_0 + \sum A_\omega \exp[i(Kx - \omega t)]$$

Where A_0 is the macroscopic parameter in the absence of vibrations. A is the amplitude of vibrations, ω is the angular frequently of vibration and K is the respective wave number depending on the amplitudes for other frequencies.

Plasma Oscillations.

In analyzing the oscillations which are due to the motion of charged particles in a plasma or weakly ionized gas, in the simplest case of homogenous plasma, and in the absence of external fields, there are two kinds of the natural plasma oscillations since plasma has been two species of charged particles. This kinds of oscillations differs considerably since the electrons and ions responsible for them differ greatly in mass [2].

Studying the high frequency oscillations of the homogeneous plasma, these oscillations are due to electrons motion. They are referred to as plasma waves. Because of their large mass, the ions are not involved in these oscillations and when analyzing plasma waves, one could assume the ions to be at rest then charges are uniformly distributed over the gas volume.

The dispersion relation could be derived from plasma waves from the continuity equation,

$$\frac{\partial N}{\partial t} + div(N_\omega) = 0 \qquad (7)$$

The Euler equation

$$\frac{\partial w}{\partial t} + (\omega \nabla)\omega + \frac{grad\rho}{\rho} - \frac{F}{m} = 0 \qquad (8)$$

And the adiabatic equation, $PV^\gamma =$ constant for waves.

Also, the electric field produced by the motion of the election owing to disturbance of the quassineutrality of plasma must be taken into account.

Introducing the electric field term into Euler equation and the electric field strength will be given by Poisson's equation

$$divE = -\nabla^2\phi = 4\pi e(N_i - N_e) \qquad .(9)$$

Similar to the derivation of the dispersion relation for acoustic oscillations and assuming also that the macroscopic parameters of the oscillating plasma can be written in the form of

$A = A_0 + A\exp[i(Kx - \omega t)]$,and when there is no oscillation, the mean velocity ω of electrons and the electric field strength E are zero and obtained

$$\begin{aligned}
&-i\omega N_e' + iKN_o\omega' = 0 \\
&-i\omega\omega' + \frac{iKP'}{mN_o} + \frac{eE'}{m} = 0 \qquad (10) \\
&\frac{P'}{P_o} = \frac{rN_e}{N_o} \\
&iKE' = -4\pi eN_o
\end{aligned}$$

K and ω are the wave number and the frequency of the plasma oscillations. N_0 is the mean density of charged particles $P_o = N_0 m(V_x^2)$ is the electron gas pressure in the absence of oscillations, m is the electron mass, V_x is the electron velocity component in the direction of oscillations and the angular bracket denotes averaging over the electron velocities. The quantities N_e', ω', P' and E' in equation (10) are the oscillations amplitudes of the electron density, mean velocity, pressure and electric field strength respectively.

On the other hand the time derivative of Ampere's law and the force equation can be combined to give an equation for the fields. [4].

$$\frac{\partial^2 \mathrm{E}}{\partial t^2}+\left[\frac{4\pi e^2 n_o}{m}\right]\mathrm{E}-\frac{1}{m}\left[\frac{\partial p}{\partial n}\right]_0 \nabla(\nabla.\mathrm{E})=c\nabla\times\frac{\partial B}{\partial t}.) \quad (11)$$

Combining the continuity and force equation, a wave equation for the density functions resulted as

$$\frac{\partial^2 n}{\partial t^2}+\left[\frac{4\pi e^2 n_o}{m}\right]n-\frac{1}{m}\left[\frac{\partial p}{\partial n}\right]_0 \nabla^0 n=0 \quad (12)$$

The structures on left hand side of equations (11) and (12) are identical.

For $\frac{\partial B}{\partial t}=0$, having excluded static fields, it could be concluded that $B=0$ is a possibility. If $\frac{\partial B}{\partial t}=0$,then faraday's law implies

$$\nabla\times\mathrm{E}=0$$

Hence, E is a longitudinal field derivable from a scalar potential.

If the pressure term in equation (12) is neglected, it could be found that the density, velocity, and electric field all oscillate with the plasma frequently ω_p

$$\omega_p^2=\frac{4\pi n_0 e^2}{m}. \quad (13)$$

If the pressure term is included, a dispersion relation is obtained for the frequency:

$$\omega^2=\omega_p^2+\frac{1}{m}\left[\frac{\delta p}{\delta n}\right]_0 k^2 \quad (14)$$

Dispersion Relations

Considering the case in which no external field exists and neglecting the effect of collisions between the electrons and heavier particles, starting from the momentum equation for the electrons [6]

$$N-m\left[\frac{\delta V_i}{\delta t}+N_j\right]\frac{\delta V_i}{\delta X_j}+\frac{\delta\psi_{ij}}{\delta X_j}-N_q E_i=0 \quad (15)$$

Writing ψ_{ij} as some scalar $P-\sigma_j$ where the effect of the anisotropy can be included in the particular form of P,hence the momentum equation can be written as

$$\frac{\partial V_i}{\partial t}+(V_i)\frac{\partial V_i}{\partial x_i}==\frac{-e}{m}E_i-\frac{1}{Nm}\frac{\partial P}{\partial x_i} \quad (16)$$

Equation (16) is nonlinear and in order to proceed, it is very important to make it linear.

Assuming that the wave phenomena or oscillations are adequately described as small perturbations from an equilibrium position, that is, electron velocity $\langle V_i\rangle$ and electron number density N can be written as

$$\langle\overline{V_i}\rangle=\overline{C_i}+\overline{V_i} \quad (17)$$

$$N=N_o+\overline{N_i} \quad (18)$$

Where $\overline{C}_i$ and N_0 are constant average values while V_i and N represent all perturbations from equilibrium. In the linearization process, products of these small quantities are neglected. From the pressure gradient term in equation (16) it could be written as

$$\frac{1}{m}\frac{\partial p}{\partial x_1}=\frac{1}{m}\frac{\partial p}{\partial\overline{N}}\frac{\partial\overline{N}}{\partial x_i}=a^2\frac{\partial\overline{N}}{\partial x_i} \quad (19)$$

Where *Where* $a=\left[\frac{I}{m}\frac{\partial p}{\partial\overline{N}}\right]^{\frac{1}{2}}. \quad (20)$

If the plasma is assumed to be Isothermal then

$P=N_0KT_0=\left(N_0+\overline{N}\right)KT_0$, hence,

$$a^2 = \frac{KT}{m} \tag{21}$$

When plasma oscillations behave more likely adiabatically, the adiabatic law can be assumed [4] and [7]

$$\frac{P}{P_0} = \left[\frac{N}{N_0}\right]^y \tag{22}$$

Where $P_0 = N_0 KT_0 = Constatnt$ (23)

, and T_0 is the constant average part of the electron temperature [1] using equations (22) and (23) and having known that $\overline{N}$ << N ,it is found that

$$\frac{\partial p}{\partial N} = N_0 KT_0 \frac{\partial}{\partial \overline{N}} \frac{\partial}{\partial \overline{N}} \left[\frac{N_0 + \overline{N}}{N_0}\right]^\gamma = N_0 KT_0 \frac{\partial}{\partial N}\left[1 + \frac{\partial \overline{N}}{N_0}\right]. \tag{24}$$

$$or \quad \frac{\partial p}{\partial \overline{N}} = \gamma KT_0 \tag{25}$$

$$So \quad that \quad a^2 = \frac{\gamma Kt_o}{m}. \tag{26}$$

It is noted that equation (26) is different from equation (21) only in the constant γ. for an ideal gas with three degrees of freedom, it is found that γ=5/3. due to the anisotropy nature of the plasma, in consideration, one degree of freedom is taken into consideration in which γ=3

In any event, equation (19) will be used in momentum equation where velocity 'a' is of order of the thermal velocity. Substituting equation (17), (18) and (19) into equation into equation (16) and neglecting terms involving the products of the small perturbations, yields.

$$\frac{\partial \overline{V}_i}{\partial t} + Cj\frac{\partial \overline{V}_i}{\partial xj} = \frac{-e}{m} Ei - a^2 \frac{\partial \overline{N}}{\partial x_1} \tag{27}$$

Considering plasma whose electron has no constant average velocity, then $C_1 = 0{,}50$ that the only electron motion contributing (V_i) is the small oscillation motions

Hence in this case equation (27) can be written as $\frac{\partial \overline{V} i}{\partial t} = -\frac{e}{m} E_1 - \frac{a^2}{N_0}\frac{\partial \overline{N}}{\partial x}$ (28)

The continuity equation for this electron is $\frac{\partial N}{\partial t} + (N - \langle V_i \rangle ,i= 0)$

Which can be written in linearized form using equation (17) and (18) with $C_i = 0$ as

$$\frac{\partial N}{\partial t} + (\overline{N} \ \langle V_i \rangle)_{i,i} = 0 \tag{29}$$

From Maxwell's equation

$$E_{ij} E_{kj} = -B_i \tag{30}$$

$$E_{ijk} B_{kj} = \mu_0 \overline{j}_i + \mu_0 \in_0 E_i \tag{31}$$

Where the current density $\overline{j}_i$ is given, after linearing by

$$\bar{j}_i = -N_0 e \bar{V}_i \tag{32}$$

Equations (28) to equation (32) represent a total of thirteen scalar equation with thirteen unknowns E_i, B_i, $\bar{j}$, V_i and N of the form $\ell^i\left(K \propto_q x_q - wt\right)$ the equations (28) to (32) can be written respectively as.

$$-iw\hat{V}_1 = -\frac{e}{m_i}E_i - iK\beta, a^2 \frac{\bar{N}}{N_0} \tag{33}$$

$$iw\bar{N} + ik \propto_i No\hat{V}_i = 0 \tag{34}$$

$$iKE_i jK \propto jE_k = iwB \tag{35}$$

$$iKE_i jKB_k = \mu_0 ji - iw\mu_0 \in_0 E_i \tag{36}$$

$$j_i = -N_0 e\hat{V}_i \tag{37}$$

Equation 35 can be written as

$$B_k = \frac{k}{w} \in_{krs} \propto_r ts) \tag{38}$$

Substituting equations (38) and (37) into (36) and using

$C^2 = \dfrac{1}{\mu_0 \in_0}$, we have

$$\frac{iK^2}{\omega} EijKE_{krs} \propto j \propto r\text{E}s = -\mu_0 N_0 e\hat{V}_i - i\omega\mu_0 \in_0 E_i$$

or $$E_{kj}E_{krs} \propto sEs = i\omega\mu_0 N_0 e\hat{V}_i - \frac{\omega^2}{K^2C^2}E_i$$

Hence

$$\left(\sigma_{ir} - \sigma_{is}\sigma_{jr}\right) \propto j_{\propto r}E_s = \frac{i\omega\mu_0 N_0 e}{K^2}\hat{V}i - \frac{\omega^2}{K^2 l^2}E_i$$

Or since $\propto_i \propto_j = 1$

$$\hat{V}i = \frac{K^2}{i\omega\mu_0 N_0 e}\left[\propto_i \propto_j - \sigma_{ij}\left[1 - \frac{\omega^2}{K^2C^2}\right]\right]E_j. \tag{39}$$

From equation 34,

$$\hat{N} = \frac{K}{\omega}N_0 \propto_j \bar{V}_j \tag{40}$$

Substituting equation (40) into equation (33) the resulting equation is

$\hat{V} = \dfrac{ie}{m\omega}E_i + \dfrac{K^2a^2}{\omega^2} \propto_i \propto_j \hat{V}_j$,from which

$$\frac{-ie}{m\omega}E_i = \left[\sigma_{ij} - \frac{K^2a^2}{\omega^2} \propto_i \propto_j\right]\hat{V}_j \tag{41}$$

,is obtained substituting equation (40) into equation (33) it result to:

$$\frac{-ie}{m\omega}E_i = \left[\sigma_{ij} - \frac{K^2a^2}{\omega^2} \propto_i \propto_j\right]\hat{V}_j \tag{42}$$

Substituting equation (39) into equation (41) it leads to.

$$\frac{N_\ell \ell^2}{m\in_0}.\frac{\mu_0 \in_0 \text{E}_i}{K^2} = \left[\sigma_{ij} - \frac{K^2a^2}{\omega^2} \propto_i \propto_j\right]\left[\propto_i \propto_j - \delta_{jk}\left(1 - \frac{\omega^2}{K^2C^2}\right]E_k$$

$$\frac{\pi^2}{K^2C^2}Ei = \left[\propto_i \propto_j - \sigma_{ik}\left(1 - \frac{\omega^2}{K^2C^2}\right) - \frac{K^2a^2}{\omega^2}\propto_i \propto_k + \propto_i \propto_k \frac{K_2 a_2}{\omega_2}\left(1 - \frac{\omega^2}{K_2C_2}\right)\right]E_k$$

$$= \left[\propto_i \propto_k \left(1 - \frac{a^2}{C^2}\right) - \sigma_{ik}\left(1 - \frac{w^2}{K^2C^2}\right)\right]E_k$$

From which

$$\left[\sigma_{ik}\left(1 - \frac{w^2}{K^2C^2} + \frac{\pi^2}{K^2C^2}\right) - \propto_i \propto_k \left(1 - \frac{a^2}{C^2}\right)\right]E_k = 0 \qquad (43)$$

If it is assumed that the wave is propagating in the X_Z $-$direction so that $ijk = (0,0,1)$

$$\propto_i \propto_k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$Equation(43)$

$$\begin{pmatrix} 1 - \frac{W^2}{K^2C^2} + \frac{\pi^2}{K^2C^2} & 0 & 0 \\ 0 & 1 - \frac{W^2}{K^2C^2} + \frac{\pi^2}{K^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}. \qquad (44)$$

From the matrix, we obtain

$$\left(1 - \frac{\omega^2}{K^2C^2} + \frac{\pi^2}{K^2C^2}\right)E_i = 0 \qquad (45)$$

$$\left(1 - \frac{\omega^2}{K^2C^2} + \frac{\pi^2}{K^2C^2}\right)E_2 = 0 \qquad (46)$$

$$\left(1 - \frac{\omega^2}{K^2C^2} + \frac{\pi^2}{K^2C^2}\right)E_3 = 0 \qquad (47)$$

For the fields $E_1, E_2 and E_3$ to exist, the coefficients in each case must vanish from equation (45) and (46) it is observed that the dispersion equation for the Transverse field $E_1 and E_2$ is

$$\omega^2 = \pi^2 + K^2C^2 \qquad (48)$$

However, from the equation (48) it is also observed from that longitudinal wave in which the field E is in the direction of propagation and that the dispersion equation for this wave is given by

$$\omega^2 = \pi^2 + K^2a^2 \qquad (49)$$

Hence, it is observed that the dispersion equation is of the same form as (48 and 49) but in transverse waves, the thermal speed 'a' replace the speed of light C.

In view of the fact that Maxwell's equations in the treatment of wave is mathematically more INVOLVING although richer, when considering only longitudinal modes of propagation, it is sufficient to use POISSON'S equation rather that the full Maxwell's equation in the wave treatment.

Considering electron charge e, mass m, density n(x,t) and velocity v(x,t), the dynamic equation for the electron field are:

$$\frac{\partial n}{\partial t}+\nabla.(nV)=0.$$

$$\frac{\partial V}{\partial t}-(v.\nabla)v=\frac{e}{m}\left(E\times\frac{v}{c}B\right)-\frac{1}{mn}\nabla P \quad (51)$$

The electron pressure 'p' describes the thermal kinetic energy effect and this is assumed to be scalar.
The charge current

$$\ell_e = e(n-n_0) \quad (52)$$

The current density $j = neV$ (53)

In equation (52) $'n_0'$ represent constant average value of the electron density and 'n' represent all disturbances or perturbations from equilibrium.
From the Maxwell's equation

$\nabla.E = 4\pi\rho_\ell$, hence substituting for

$\ell_e = e(n-n_0)$ it gives

$$\nabla.E = 4\pi_e(n - \mathrm{n}_o) \quad (54a)$$

$$\nabla.B = 0 \quad (54b)$$

$$\nabla\times E+\frac{1}{c}\frac{\partial B}{\partial t}=0 \quad (54c)$$

$$\nabla\times B-\frac{1}{C}\frac{\partial E}{\partial t}=\frac{4\pi_e n}{C}V \quad (54d)$$

$$\frac{\partial n}{\partial t}+n_0\nabla.V=0 \quad (55a)$$

$$\frac{\partial V}{\partial t}-\frac{e}{m}\mathrm{E}+\frac{1}{mn_0}\left(\frac{\partial P}{\partial n}\right)_0\nabla n=0 \quad (55b)$$

Equation (55a) is linearized equation of motion.

$$\nabla.E-4\pi_e n=0$$

$$\nabla\times B=-\frac{1}{C}\frac{\partial B}{\partial t}+\frac{4\pi_e\mu_0}{C}V$$

Assuming a static state for the electron, that is they are at rest, it implies that $n = n_0$ and n_0 prescience of field is observed.
Due to the initial disturbances, a static state is now departed, no linearising the equations of motion,
Equation (55b) is the homogeneous Maxwell's equations.
Equations 55a and 55b are now independent of magnetic field. Hence, $B = 0$ which now show that the solution of the force equation is purely electrostatic in nature.
The continuity equation for the electron is purely electrostatic in nature.

The continuity equation for the electron is $\frac{\partial n}{\partial t}+(\bar{n}\langle V\rangle) = 0$ (56)

Now combining the continuity and force equations, it resulted in to $\frac{\partial^2 n}{\partial t^2}+\frac{4\pi_e{}^2}{m}(n_o)\bar{n}\frac{1}{m}\left(\frac{\partial P}{\partial n}\right)\nabla^2 n=0$ (57)

Equation (57) is a wave equation for density fluctuation.

Combining the time derivative of Ampere's law and for equation:

$$\frac{\partial^2 E}{\partial t^2}+\left(\frac{4\pi e^2}{m}no\right)E-\frac{1}{m}\left(\frac{\partial P}{\partial n}\right)_0 \nabla(\nabla.E)=\left(\nabla\times\frac{\partial B}{\partial t}\right) \quad (58)$$

Substituting for $\frac{\partial B}{\partial t}=0$, it implies that $\nabla\times E=0$

Neglecting the pressure term in equation (58), the density, velocity and electric field oscillate with the plasma frequency.

$$\omega_p{}^2=\frac{4\pi n_0 e^2}{m} \quad (59)$$

Including the pressure term , the dispersion relation for the frequency is

$$\omega^2=\omega_p{}^2+\frac{1}{m}\left(\frac{\partial P}{\partial N}\right)_0 K^2 \quad (60)$$

Where K is the wave number

From the adiabatic law, $P=P_0\left(\frac{n}{n_0}\right)^\gamma$ (61)

Since one dimensional oscillation is maintained, degree of freedom on (1) is appropriate [4]. Hence, $\gamma=m+\frac{2}{m}$, here, m is degree of freedom. Substituting the degree of freedom, it results into $\gamma=1+\frac{2}{1}=3$

Hence, $\gamma=3$ for this case, therefore, $\frac{1}{m}\left(\frac{\partial P}{\partial n}\right)_0=\frac{3P_0}{mn_0}$ (62)

From the ideal gas law, $P_0=n_0KT$. the root means square velocity components in one direction parallel to the electric field is m < u$\geq KT=\frac{P_0}{mn_0}$ (63)

Hence, equation 60 is the dispersion relation and can be re-written as

$$\omega^2=\omega_p{}^2+3<u^2>K^2 \quad (64)$$

Equation (64) is valid for long wave length.

Conclusion.

The dispersion equation (64) $\omega^2=\omega_p{}^2+3<u^2>K^2$ is approximate one, valid for long wavelengths and is actually just the first two terms in expansion involving higher and higher moments of the velocity distribution of the electrons.

The dispersion equation has validity beyond the ideal gas law which was used in the derivation.

From the wave treatment of this form, although, Poisson's equation was derived from the Maxwell's equation, it is sufficient to treat a longitudinal wave completely to get the dispersion relation in plasma which is a function of density, velocity and the field effect. Treatment of waves – particles interaction involving magneto hydrodynamic equations of charged particles in arbitrary wave fields are mathematically cumbersome when using full Maxwell's equation but using Poisson's equation is the simplest way of treating such waves when propagation of special longitudinal waves in plasma is considered.

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