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Effect of Frank-Kamenetskii Parameter on the Propagation of Forward and Opposed Smouldering Combustion

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Abstract

Smouldering combustion, the slow burning process associated with porous solid have been studied by mathematical point of view. We assume that there is a perfect contact between gas and solid phases. We examine the properties of solution of the steady-state problems under certain conditions. The equations are solved analytically using asymptotic expansions. The steady-state temperature distribution and species mass fraction profiles are presented and discussed. It is discovered that the Frank-Kamenetskii number plays a crucial role in the slow burning process and the temperature is decreased and species is consumed in the spatial direction.

Keywords: Smouldering combustion, solid porous fuel, energy sink, unburnt fuel, porous matrix.

1.0 Introduction

Combustion is the exothermic oxidation of a fuel. In the case of a carbon-based compound, the products are primarily carbon dioxide, water and energy.

Smouldering phenomenon is a flameless form of combustion, deriving its heat from heterogeneous reactions occurring on the surface of a solid fuel when heated in an oxidizer environment [1]. It is of interest both as a fundamental combustion problem and as a practical fire hazard.

Smouldering is limited by the rate of oxygen-transport to the fuel's surface (see Figure 1), resulting in a slower and lower temperature reaction than flaming. Importantly, smouldering can be self-sustaining (i.e., no energy input required after ignition) when the fuel is (or is embedded in) a porous medium. Self-sustaining smouldering occurs because the solid acts as energy sink and then feeds that energy back into the unburnt fuel, creating a very energy efficient reaction [2]. Solid porous fuels such as polyurethane foam [3], cellulose [4] and charcoal are typical media that exhibit self-sustained smouldering.

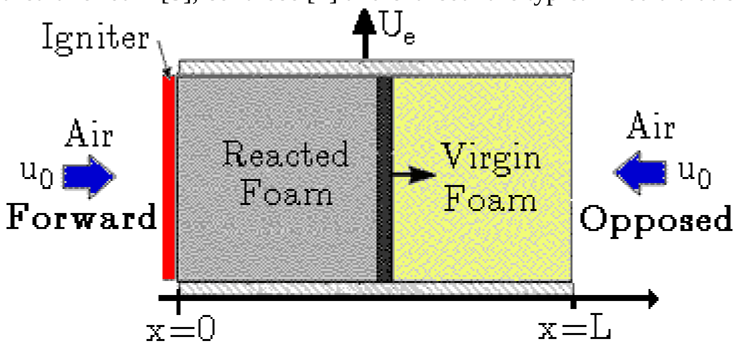

Figure 1: Computational domain for opposed and forward smouldering combustion.

While most research focuses on smouldering of solid fuels, there are several examples of combustion of a liquid fuel embedded in a porous matrix. Lagging fires occur inside porous insulating materials soaked in oils and other self-igniting

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liquids (Drysdale [5]). To enhance oil recovery, combustion fronts are initiated in petroleum reservoirs to drive oil toward extraction points (Greaves et al. [6]). The reactions involved in enhanced oil recovery through in situ combustion are described as heterogeneous gas-solid and gas-liquid between oxygen and the heavy oil residue (Sarathi [7]).
Rein et al. [8] carried out a computational study to investigate smouldering ignition and propagation in polyurethane foam. Forward and opposed smouldering configurations are examined with the numerical model and new kinetics.
In this paper, one-dimensional, steady, governing equation for smouldering combustion in a porous fuel is considered. We assume that there is a perfect contact between gas and solid phases. We consider the presure gradient to be parabolic. We examine the properties of solution under certain conditions. To simulate the flow analytically, we use asymptotics expansions.

2.0 Mathematical Model

The steady, one-dimensional governing equations for smouldering combustion in a porous fuel is given by the equation of :
Conservation of energy of solid

$$\frac{1}{\rho_s C_{ps}}\frac{d}{dx}\left(k_s\frac{dT_s}{dx}\right)+\frac{h_{gs}A_{gs}}{\rho_s C_{ps}V}\left(T_g-T_s\right)-\frac{U_e A_L}{\rho_s C_{ps}V}\left(T_s-T_0\right)-\frac{\rho_0\Delta hA}{\rho_s C_{ps}}e^{-\frac{E}{RT_s}}=0 \qquad (1)$$

Conservation of energy of gas

$$\frac{1}{\rho_g C_{pg}}\frac{d}{dx}\left(\phi k_g\frac{dT_g}{dx}\right)+\frac{d}{dx}\left(\frac{K}{\mu}\frac{\partial P}{\partial x}\left(T_g-T_0\right)\right)-\frac{h_{gs}A_{gs}}{\rho_g C_{pg}V}\left(T_g-T_s\right)=0 \qquad (2)$$

Conservation of gas species: Oxygen

$$\frac{1}{\rho_g}\frac{d}{dx}\left(\rho_g D\frac{dy_{o_2}}{dx}\right)+\frac{d}{dx}\left(\frac{K_x}{\mu}\frac{\partial P}{\partial x}y_{o_2}\right)-\frac{\rho_0 v_{o_2}A}{\phi\rho_g}e^{-\frac{E}{RT_s}}=0 \qquad (3)$$

Smouldering product

$$\frac{1}{\rho_g}\frac{d}{dx}\left(\rho_g D\frac{dy_{gp}}{dx}\right)+\frac{d}{dx}\left(\frac{K_x}{\mu}\frac{\partial P}{\partial x}y_{gp}\right)+\frac{\rho_0 v_{gp}A}{\phi\rho_g}e^{-\frac{E}{RT_s}}=0 \qquad (4)$$

The boundary conditions were formulated as follows:
Boundary conditions:

$$\left.\begin{aligned}&T_g\big|_{x=0}=T_0,\quad T_g\big|_{x=L}=T_0\\&T_s\big|_{x=0}=T_0,\quad T_s\big|_{x=L}=T_0\end{aligned}\right\} \qquad (5)$$

$$\left.\begin{aligned}&y_{o_2}\big|_{x=0}=y_0,\quad y_{o_2}\big|_{x=L}=0\\&y_{gp}\big|_{x=0}=0,\quad y_{gp}\big|_{x=L}=0\end{aligned}\right\}, \qquad (6)$$

where
$\frac{A_{gs}}{V}$ is the ratio of surface area between gas and solid to volume, $\frac{A_L}{V}$ is the ratio of lateral area to volume, E is activation energy, R is the perfect gas constant, L is sample length, k is thermal conductivity, D is the diffusion coefficient, Δh is the enthalpy of reaction, C is specific heat, U_e is the global heat-loss coefficient to exterior, T is temperature, y is the mass fraction of gas species, x is position, h_{gs} is the heat transfer coefficient between gas and solid, K is permeability of the medium, μ is dynamic viscosity, P is pressure, ρ is density, ϕ is the porosity of the medium.
We assume that there is a perfect contact between gas and solid phases so that one can make the hypothesis of local thermal equilibrium between the phases:

$$T_g=T_s=T \qquad (7)$$

Adding (1) and (2), we obtain

$$\frac{d}{dx}\left(\lambda\frac{dT}{dx}\right)+\frac{d}{dx}\left(\frac{K}{\mu}\frac{\partial P}{\partial x}\left(T-T_0\right)\right)-\frac{U_e A_L}{\rho_s C_{ps} V}\left(T-T_0\right)-\frac{\rho_0 \Delta h A}{\rho_s C_{ps}}e^{-\frac{E}{RT}}=0, \tag{8}$$

where

$\lambda=\frac{k_s}{\rho_s C_{ps}}+\frac{\phi k_g}{\rho_g C_{pg}}$ is the overall thermal conductivity.

3.0 Method of Solution

Here, we make the additional assumptions that $\rho_g,\ \phi,\ \lambda,\ D,\ K,\ K_x$ and μ are constant, and we consider the pressure gradient to be parabolic i.e.

$$\frac{\partial P}{\partial x}=f(x)=\frac{x}{L}\left(1-\frac{x}{L}\right) \tag{9}$$

These assumptions could be relaxed in the future.
By introducing the following dimensionless variables:

$$\theta=\frac{E}{RT_o^2}(T-T_o),\quad Y=\frac{y_{o_2}}{y_{o_2}^0},\quad Z=\frac{y_{gp}}{y_{gp}^0},\quad \in=\frac{RT_0}{E},\quad x'=\frac{x}{L}, \tag{10}$$

Equations (3), (4) and (8) after dropping prime become

$$\lambda\frac{d^2\theta}{dx^2}+k_1x(1-x)\frac{d\theta}{dx}+k_1(1-2x)\theta-\alpha\theta-\delta e^{\frac{\theta}{1+\in\theta}}=0 \tag{11}$$

$$D\frac{d^2Y}{dx^2}+k_2x(1-x)\frac{dY}{dx}+k_2(1-2x)Y-\beta e^{\frac{\theta}{1+\in\theta}}=0 \tag{12}$$

$$D\frac{d^2Z}{dx^2}+k_2x(1-x)\frac{dZ}{dx}+k_2(1-2x)Z+\sigma e^{\frac{\theta}{1+\in\theta}}=0 \tag{13}$$

together with the boundary conditions

$$\left.\begin{array}{l}\theta(0)=0,\quad \theta(1)=0\\ Y(0)=Y_0,\quad Y(1)=0\\ Z(0)=0,\quad Z(1)=0\end{array}\right\}, \tag{14}$$

where

$$k_1=\frac{\phi KL}{\mu},\quad \delta=\frac{\Delta h\rho_0 AL^2.e^{-\frac{E}{RT_0}}}{\in T_0\rho_s C_{ps}},\quad \alpha=\frac{U_e A_L L^2}{\rho_s C_{ps}V},\quad k_2=\frac{K_x L}{\mu},\quad \beta=\frac{\rho_0 Av_{o_2}L^2.e^{-\frac{E}{RT_0}}}{\phi\rho_g y_{o_2}^0},\quad \sigma=\frac{\rho_0 Av_{gp}L^2.e^{-\frac{E}{RT_0}}}{\phi\rho_g y_{gp}^0}$$

3.1 Existence and Uniqueness of Solution

Theorem 1

Let $\alpha=0$, $k_1=k_2=k$, $D=\lambda=1$ and $\beta=\sigma=\delta$. Then there exists a unique solution of problem (11), (12) and (13) satisfy (14).

Proof:

Let $\alpha = 0$, $k_1 = k_2 = k$, $D_1 = \lambda_1 = 1$, $\beta = \sigma = \delta$ and $\phi = \left(Z + \frac{1}{2}(\theta + Y) \right)$.

Then (11) - (14) become

$$\frac{d^2\phi}{dx^2} + kx(1-x)\frac{d\phi}{dx} + k(1-2x)\phi = 0 \tag{15}$$

$$\phi(0) = \frac{1}{2}Y_0, \quad \phi(1) = 0 \tag{16}$$

Using Frobenius method, we obtain the solution of problem (15) and (16) in series form as (using the first few terms of the series)

$$\phi(x) = \frac{1}{2}Y_0\left(1 + x - \frac{1}{2}kx^2 + \frac{1}{8}(k^2 + 2k)x^4\right) - \frac{1}{2}\frac{Y_0\left(2 - \frac{1}{2}k + \frac{1}{8}(k^2 + 2k)\right)}{\left(1 - \frac{1}{12}k + \frac{1}{15}k^2\right)} \left(x - \frac{1}{3}kx^3 + \frac{1}{4}kx^4 + \frac{1}{15}k^2x^5\right) \tag{17}$$

Then, we can write

$$\theta(x) = 2(\phi(x) - Z(x)) - Y(x) \tag{18}$$

$$Y(x) = 2(\phi(x) - Z(x)) - \theta(x) \tag{19}$$

$$Z(x) = \phi(x) - \frac{1}{2}(\theta(x) + Y(x)) \tag{20}$$

Hence, there exists a unique solution of problem (11) - (14). This completes the proof.

3.1 Properties of Solution

Theorem 2

Let $\alpha = 0$ and $k_1 = 0$ in (11). Then $\theta(x)$ is symmetric about $x = \frac{1}{2}$.

Proof: Let $\alpha = 0$ and $k_1 = 0$ in (11). We obtain

$$\frac{d^2\theta(x)}{dx^2} - \delta_1 \exp\left(\frac{\theta(x)}{1 + \in \theta(x)}\right) = 0, \qquad \phi(0) = 0, \qquad \phi(1) = 0$$

Let $y = 2x - 1$

Then

$$\frac{d^2}{dx^2} = 4\frac{d^2}{dy^2}$$

So the problem becomes

$$\frac{d^2\theta(y)}{dy^2} - \frac{\delta_1}{4}\exp\left(\frac{\theta(y)}{1 + \in \theta(y)}\right) = 0, \qquad \theta(-1) = 0, \qquad \theta(1) = 0$$

It suffices to show that $\theta(-y) = \theta(y)$.

Replace y by $-y$. We obtain

$$\frac{d^2\theta(-y)}{d(-y)^2} - \frac{\delta_1}{4}\exp\left(\frac{\theta(-y)}{1 + \in \theta(-y)}\right) = 0$$

Hence θ is symmetric about $y = 0$ i.e. θ is symmetric about $x = \frac{1}{2}$. This completes the proof.

Theorem 3

Let $\alpha = 0$ and $k_1 = 0$ in (11). Then $\theta'\left(\frac{1}{2}\right) = 0$.

Proof: Let $\alpha = 0$ and $k_1 = 0$ in (11). We obtain

$$\frac{d^2\theta(x)}{dx^2} - \delta_1 \exp\left(\frac{\theta(x)}{1+\in \theta(x)}\right) = 0, \qquad \theta(0) = 0, \qquad \theta(1) = 0$$

Since $\theta(x)$ is symmetric about $x = \frac{1}{2}$. Then $\theta'\left(\frac{1}{2}\right) = 0$. This completes the proof.

3.3 Analytical Solution

Here, we consider equations (11) - (14) when $D = \lambda = 1$. Ayeni [9] has shown that $\exp\left(\frac{\theta}{1+\in \theta}\right)$ can be approximated as $1 + (e-2)\theta + \theta^2$. In this paper we are going to take an approximation of the form

$$\exp\left(\frac{\theta}{1+\in \theta}\right) \approx 1 + (e-2)\theta \qquad (21)$$

Using the asymptotic expansion

$$\theta = \theta_0 + \in \theta_1 + \in^2 \theta_2 + h.o.t. \qquad (22)$$

$$Y = Y_0 + \in Y_1 + \in^2 Y_2 + h.o.t. \qquad (23)$$

$$Z = Z_0 + \in Z_1 + \in^2 Z_2 + h.o.t., \qquad (24)$$

where $h.o.t.$ read "higher order terms in $\in$. In our analysis we are interested only in the first two terms.

Let

$$k_1 = \in k_0 \qquad (25)$$

$$k_2 = \in q_0 \qquad (26)$$

and equate the powers of $\in$, we have the following set of non-homogeneous boundary value problems.

$O(1)$:

$$\frac{d^2\theta_0}{dx^2} - p\theta_0 - \delta = 0 \qquad (27)$$

$$\theta_0(0) = 0, \qquad \theta_0(1) = 0$$

$$\frac{d^2 Y_0}{dx^2} - \beta_0\theta_0 - \beta = 0 \qquad (28)$$

$$Y_0(0) = Y_0, \qquad Y_0(1) = 0$$

$$\frac{d^2 Z_0}{dx^2} + \sigma_0\theta_0 + \sigma = 0 \qquad (29)$$

$$Z_0(0) = 0, \qquad Z_0(1) = 0$$

$O(\in)$:

$$\frac{d^2\theta_1}{dx^2} + k_0 x(1-x)\frac{d\theta_0}{dx} + k_0(1-2x)\theta_0 - p\theta_1 = 0 \qquad (30)$$

$$\theta_1(0)=0, \qquad \theta_1(1)=0$$

$$\frac{d^2Y_1}{dx^2}+q_0x(1-x)\frac{dY_0}{dx}+q_0(1-2x)Y_0-\beta_0\theta_1=0 \qquad (31)$$

$$Y_1(0)=0, \qquad Y_1(1)=0$$

$$\frac{d^2Z_1}{dx^2}+q_0x(1-x)\frac{dZ_0}{dx}+q_0(1-2x)Z_0+\sigma_0\theta_1=0 \qquad (32)$$

$$Z_1(0)=0, \qquad Z_1(1)=0,$$

where $p=(\alpha+\delta(e-2))$, $\beta_0=\beta(e-2)$ and $\sigma_0=\sigma(e-2)$

We obtain the solution of (27) - (32) respectively as

$$\theta_0(x)=c_1e^{\sqrt{p}\,x}+c_2e^{-\sqrt{p}\,x}-\frac{\delta}{p} \qquad (33)$$

$$Y_0(x)=\frac{\beta_0}{p}\left(c_1e^{\sqrt{p}\,x}+c_2e^{-\sqrt{p}\,x}-\frac{\delta}{2}x^2\right)+\frac{\beta}{2}x^2+c_3x+c_4 \qquad (34)$$

$$Z_0(x)=-\frac{\sigma_0}{p}\left(c_1e^{\sqrt{p}\,x}+c_2e^{-\sqrt{p}\,x}-\frac{\delta}{2}x^2\right)-\frac{\sigma}{2}x^2+c_5x+c_6 \qquad (35)$$

$$\theta_1(x)=\frac{1}{6}\frac{1}{p^2}\left(k_0\left(\begin{array}{l}\left(\begin{array}{l}-\frac{3}{2}x(1-x)p^{\frac{3}{2}}-\frac{3}{2}x^2p^2+x^3p^2+\frac{3}{4}\sqrt{p}+\\ \frac{3}{4}p-\frac{3}{2}xp\end{array}\right)c_1e^{2\sqrt{p}\,x}\\ -6\delta(1-2x)e^{\sqrt{p}\,x}+\left(\begin{array}{l}\frac{3}{2}x(1-x)p^{\frac{3}{2}}-\frac{3}{2}xp-\frac{3}{4}\sqrt{p}\\ +\frac{3}{4}p-\frac{3}{2}x^2p^2+x^3p^2\end{array}\right)c_2\end{array}\right)e^{-\sqrt{p}\,x}\right) \qquad (36)$$

$$+c_7e^{\sqrt{p}\,x}+c_8e^{-\sqrt{p}\,x}$$

$$Y_1(x)=\frac{1}{24}\frac{1}{p^{\frac{5}{2}}}\left(\begin{array}{l} -6\beta_0k_0c_2e^{-\sqrt{p}\,x}\left(\sqrt{p}x+2\right)-6\beta_0k_0c_2e^{-\sqrt{p}\,x}\left(px^2+4\sqrt{p}x+6\right)- \\ 6\beta_0k_0c_1e^{\sqrt{p}\,x}\left(\sqrt{p}x-2\right)+4\beta_0k_0c_2e^{-\sqrt{p}\,x}\left(\begin{array}{l} p^{\frac{3}{2}}x^3+6px^2+ \\ 18\sqrt{p}x+24\end{array}\right)+ \\ 4\beta_0k_0c_1e^{\sqrt{p}\,x}\left(\begin{array}{l} p^{\frac{3}{2}}x^3-6px^2+ \\ 18\sqrt{p}x-24\end{array}\right)-6\beta_0k_0c_1\sqrt{p}e^{\sqrt{p}\,x}\left(\begin{array}{l} px^2- \\ 4\sqrt{p}x+6\end{array}\right) \\ -6\beta_0k_0c_2\sqrt{p}e^{-\sqrt{p}\,x}\left(\begin{array}{l} px^2+ \\ 4\sqrt{p}x+6\end{array}\right)+24\beta_0q_0c_1e^{\sqrt{p}\,x}\left(\begin{array}{l} px^2- \\ 4\sqrt{p}x+6\end{array}\right)+ \\ 24\beta_0q_0c_2\sqrt{p}e^{-\sqrt{p}\,x}\left(\sqrt{p}x+2\right)+6\beta_0k_0c_1e^{\sqrt{p}\,x}\left(px^2-4\sqrt{p}x+6\right) \\ -6\beta_0k_0c_1\sqrt{p}e^{\sqrt{p}\,x}\left(\sqrt{p}x-2\right)-24\beta_0q_0c_2e^{-\sqrt{p}\,x}\left(px^2+4\sqrt{p}x+6\right) \\ +6\beta_0k_0c_2\sqrt{p}e^{-\sqrt{p}\,x}\left(\sqrt{p}x+2\right)-24\beta_0q_0c_1\sqrt{p}e^{\sqrt{p}\,x}\left(\sqrt{p}x-2\right)+ \\ 48\beta_0q_0c_1e^{\sqrt{p}\,x}\left(\sqrt{p}x-2\right)+48\beta_0q_0c_2e^{-\sqrt{p}\,x}\left(\sqrt{p}x+2\right)+ \\ 4\beta_0k_0\delta\sqrt{p}x^2(2x-3)-\beta_0q_0\delta p^{\frac{3}{2}}x^4\left(\frac{12}{5}x-3\right)-24\beta_0q_0c_2\sqrt{p}e^{-\sqrt{p}\,x} \\ -24\beta_0q_0c_1\sqrt{p}e^{\sqrt{p}\,x}+3\beta_0k_0c_2e^{-\sqrt{p}\,x}\left(\sqrt{p}-1\right)+3\beta_0k_0c_1e^{\sqrt{p}\,x}\left(\sqrt{p}+1\right) \\ -q_0\beta\,p^{\frac{5}{2}}x^4\left(3-\frac{12}{5}x\right)-2q_0c_3p^{\frac{5}{2}}x^3(4-3x)-4q_0c_4p^{\frac{5}{2}}x^2(3-2x)+ \\ 24\beta_0c_7p^{\frac{3}{2}}e^{\sqrt{p}\,x}+24\beta_0c_8p^{\frac{3}{2}}e^{-\sqrt{p}\,x} \end{array}\right)$$

$$+c_9x+c_{10} \tag{37}$$

$$Z_1(x) = -\frac{1}{24}\frac{1}{p^{\frac{5}{2}}}\left(\begin{array}{l} -6\sigma_0 k_0 c_2 e^{-\sqrt{p}x}\left(\sqrt{p}x+2\right) - 6\sigma_0 k_0 c_2 e^{-\sqrt{p}x}\left(px^2+4\sqrt{p}x+6\right) - \\ 6\sigma_0 k_0 c_1 e^{\sqrt{p}x}\left(\sqrt{p}x-2\right) + 4\sigma_0 k_0 c_2 e^{-\sqrt{p}x}\left(\begin{array}{l} p^{\frac{3}{2}}x^3+6px^2+ \\ 18\sqrt{p}x+24 \end{array}\right) + \\ 4\sigma_0 k_0 c_1 e^{\sqrt{p}x}\left(\begin{array}{l} p^{\frac{3}{2}}x^3-6px^2+ \\ 18\sqrt{p}x-24 \end{array}\right) - 6\sigma_0 k_0 c_1 \sqrt{p}e^{\sqrt{p}x}\left(\begin{array}{l} px^2- \\ 4\sqrt{p}x+6 \end{array}\right) \\ -6\sigma_0 k_0 c_2 \sqrt{p}e^{-\sqrt{p}x}\left(\begin{array}{l} px^2+ \\ 4\sqrt{p}x+6 \end{array}\right) + 24\sigma_0 q_0 c_1 e^{\sqrt{p}x}\left(\begin{array}{l} px^2- \\ 4\sqrt{p}x+6 \end{array}\right) + \\ 24\sigma_0 q_0 c_2 \sqrt{p}e^{-\sqrt{p}x}\left(\sqrt{p}x+2\right) + 6\sigma_0 k_0 c_1 e^{\sqrt{p}x}\left(px^2-4\sqrt{p}x+6\right) \\ -6\sigma_0 k_0 c_1 \sqrt{p}e^{\sqrt{p}x}\left(\sqrt{p}x-2\right) - 24\sigma_0 q_0 c_2 e^{-\sqrt{p}x}\left(px^2+4\sqrt{p}x+6\right) \\ +6\sigma_0 k_0 c_2 \sqrt{p}e^{-\sqrt{p}x}\left(\sqrt{p}x+2\right) - 24\sigma_0 q_0 c_1 \sqrt{p}e^{\sqrt{p}x}\left(\sqrt{p}x-2\right) + \\ 48\sigma_0 q_0 c_1 e^{\sqrt{p}x}\left(\sqrt{p}x-2\right) + 48\sigma_0 q_0 c_2 e^{-\sqrt{p}x}\left(\sqrt{p}x+2\right) + \\ 4\sigma_0 k_0 \delta\sqrt{p}x^2(2x-3) - \sigma_0 q_0 \delta p^{\frac{3}{2}}x^4\left(\frac{12}{5}x-3\right) - 24\sigma_0 q_0 c_2 \\ \sqrt{p}e^{-\sqrt{p}x} - 24\sigma_0 q_0 c_1 \sqrt{p}e^{\sqrt{p}x} + 3\sigma_0 k_0 c_2 e^{-\sqrt{p}x}\left(\sqrt{p}-1\right) + \\ 3\sigma_0 k_0 c_1 e^{\sqrt{p}x}\left(\sqrt{p}+1\right) - q_0\sigma\, p^{\frac{5}{2}}x^4\left(3-\frac{12}{5}x\right) + 2q_0 c_5 p^{\frac{5}{2}}x^3 \\ (4-3x) + 4q_0 c_6 p^{\frac{5}{2}}x^2(3-2x) + 24\sigma_0 c_7 p^{\frac{3}{2}}e^{\sqrt{p}x} + \\ 24\sigma_0 c_8 p^{\frac{3}{2}}e^{-\sqrt{p}x} \end{array}\right)$$

$$+c_{11}x + c_{12} \tag{38}$$

Therefore, we obtain

$$\theta(x) = solution\ (3.25) + \in\ multiply\ solution\ (3.28) \tag{39}$$

$$Y(x) = solution\ (3.26) + \in\ multiply\ solution\ (3.29) \tag{40}$$

$$Z(x) = solution\ (3.28) + \in\ multiply\ solution\ (3.30), \tag{41}$$

where

$$c_1 = \frac{\delta}{p}\left(\frac{1-e^{-\sqrt{p}}}{e^{\sqrt{p}}-e^{-\sqrt{p}}}\right), \quad c_2 = \frac{\delta}{p}\left(\frac{e^{\sqrt{p}}-1}{e^{\sqrt{p}}-e^{-\sqrt{p}}}\right),$$

$$c_3 = \left(\frac{\beta_0}{p}\left(c_1\left(1-e^{\sqrt{p}}\right) + c_2\left(1-e^{-\sqrt{p}}\right) + \frac{\delta}{2}\right) - Y_0 - \frac{\beta}{2}\right), \quad c_4 = \left(Y_0 - \frac{\beta_0}{p}(c_1+c_2)\right),$$

$$c_5 = \left(\frac{\sigma_0}{p} \left(c_1 \left(e^{\sqrt{p}} - 1 \right) + c_2 \left(e^{-\sqrt{p}} - 1 \right) - \frac{\delta}{2} \right) + \frac{\sigma}{2} \right), \qquad c_6 = \left(\frac{\sigma_0}{p} (c_1 + c_2) \right),$$

$$c_7 = -\frac{1}{24} \frac{k_0 e^{-\sqrt{p}} \left(3c_1 \sqrt{p} \left(e^{2\sqrt{p}} - 1 - \sqrt{p} \right) + 24\delta \left(1 + e^{\sqrt{p}} \right) - 2c_2 p (3 + p) - c_1 p e^{2\sqrt{p}} (2p + 3) \right)}{p^2 \left(e^{\sqrt{p}} - e^{-\sqrt{p}} \right)},$$

$$c_8 = \frac{1}{24} \frac{k_0 \left(3c_2 \sqrt{p}\, e^{\sqrt{p}} \left(1 - \sqrt{p} \right) + 24\delta \left(1 + e^{\sqrt{p}} \right) - 2c_1 p\, e^{\sqrt{p}} (3 + p) - c_2 e^{-\sqrt{p}} \left(3p + 3\sqrt{p} + 2p^2 \right) \right)}{p^2 \left(e^{\sqrt{p}} - e^{-\sqrt{p}} \right)}$$

$$c_9 = -\frac{1}{120} \frac{1}{p^3} \begin{pmatrix} 3q_0 \beta_0 \delta p^2 + 120\beta_0 c_7 p^2 \left(e^{\sqrt{p}} - 1 \right) + 120\beta_0 c_8 p^2 \left(e^{-\sqrt{p}} - 1 \right) - 20\beta_0 k_0 \delta p - \\ \beta_0 k_0 c_2 e^{-\sqrt{p}} \left(10p^2 - 105p - 225\sqrt{p} \right) - 120q_0 \beta_0 c_2 e^{-\sqrt{p}} \left(p + 2\sqrt{p} \right) - \\ 120q_0 \beta_0 c_1 e^{\sqrt{p}} \left(p - 2\sqrt{p} \right) - \beta_0 k_0 c_1 e^{\sqrt{p}} \left(10p^2 - 105p + 225\sqrt{p} \right) + \\ \beta_0 k_0 c_2 \left(105p - 225\sqrt{p} \right) - 120q_0 \beta_0 c_2 \left(p - 2\sqrt{p} \right) - 120q_0 \beta_0 c_1 \left(p + 2\sqrt{p} \right) + \\ \beta_0 k_0 c_1 \left(105p + 225\sqrt{p} \right) - q_0 p^3 (3\beta + 20c_4 + 10c_3) \end{pmatrix}$$

$$c_{10} = -\frac{1}{8} \frac{1}{p^{\frac{5}{2}}} \beta_0 \begin{pmatrix} k_0 c_2 \left(15 - 7\sqrt{p} \right) - 8q_0 c_2 \left(2 - \sqrt{p} \right) + 8q_0 c_1 \left(2 - \sqrt{p} \right) - k_0 c_1 \left(15 + 7\sqrt{p} \right) + \\ 8p^{\frac{3}{2}} (c_7 + c_8) \end{pmatrix}$$

$$c_{11} = \frac{1}{120} \frac{1}{p^3} \begin{pmatrix} 3q_0 \sigma_0 \delta p^2 + 120\sigma_0 c_7 p^2 \left(e^{\sqrt{p}} - 1 \right) + 120\sigma_0 c_8 p^2 \left(e^{-\sqrt{p}} - 1 \right) - 20\sigma_0 k_0 \delta p + \\ 105\sigma_0 k_0 c_1 p \left(e^{\sqrt{p}} + 1 \right) + 105\sigma_0 k_0 c_2 p \left(e^{-\sqrt{p}} + 1 \right) - 120q_0 \sigma_0 c_2 p \left(e^{-\sqrt{p}} + 1 \right) - \\ 120q_0 \sigma_0 c_1 p \left(e^{\sqrt{p}} + 1 \right) - \sigma_0 k_0 c_1 e^{\sqrt{p}} \left(10p^2 + 225\sqrt{p} \right) + \\ \sigma_0 k_0 c_2 e^{-\sqrt{p}} \left(10p^2 - 225\sqrt{p} \right) - 240q_0 \sigma_0 c_2 \sqrt{p} \left(e^{-\sqrt{p}} - 1 \right) + \\ 240q_0 \sigma_0 c_1 \sqrt{p} \left(e^{\sqrt{p}} - 1 \right) + 225\sigma_0 k_0 c_1 \sqrt{p} (c_1 - c_2) + q_0 p^3 \begin{pmatrix} -3\sigma + 20c_6 \\ +10c_5 \end{pmatrix} \end{pmatrix}$$

$$c_{12} = \frac{1}{8} \frac{1}{p^{\frac{5}{2}}} \sigma_0 \begin{pmatrix} k_0 c_2 \left(15 - 7\sqrt{p} \right) + 8q_0 c_2 \left(\sqrt{p} - 2 \right) + 8q_0 c_1 \left(\sqrt{p} + 2 \right) - k_0 c_1 \left(15 + 7\sqrt{p} \right) + \\ 8p^{\frac{3}{2}} (c_7 + c_8) \end{pmatrix}$$

4.0 Results and Discussion

The existence and uniqueness of solution of the Problem is proved by the actual solution. Also, we have shown, under certain conditions, that $\theta(x)$ is symmetric about $x = \frac{1}{2}$ and $\theta'\left(\frac{1}{2}\right) = 0$. Analytical solutions given by equations (39) - (41) are computed for the values of $\alpha = 12,\ \beta = 0.2,\ \in = 0.01,\ e = 2.718,\ k_1 = k_2 = 1$. The species mass fraction and temperature values are depicted graphically in Figures 2 - 4.

The temperature distribution behavior along the spatial direction is shown in Figure 2. Figure 2 depicts the graph of $\theta(x)$ against x for different values of δ. It is observed that the temperature decreases along spatial direction as Frank-Kamenetskii number increases. The oxygen mass fraction distribution behavior along the spatial direction is shown in Figure 3. Figure 3 depicts the graph of $Y(x)$ against x for different values of δ. It is observed that the oxygen mass fraction does not change much with increase in Frank-Kamenetskii number. The smouldering product mass fraction distribution behavior along the spatial direction is shown in Figure 4. Figure 4 depicts the graph of $Z(x)$ against x for different values of δ. It is observed that the smouldering product mass fraction does not change much with increase in Frank-Kamenetskii number.

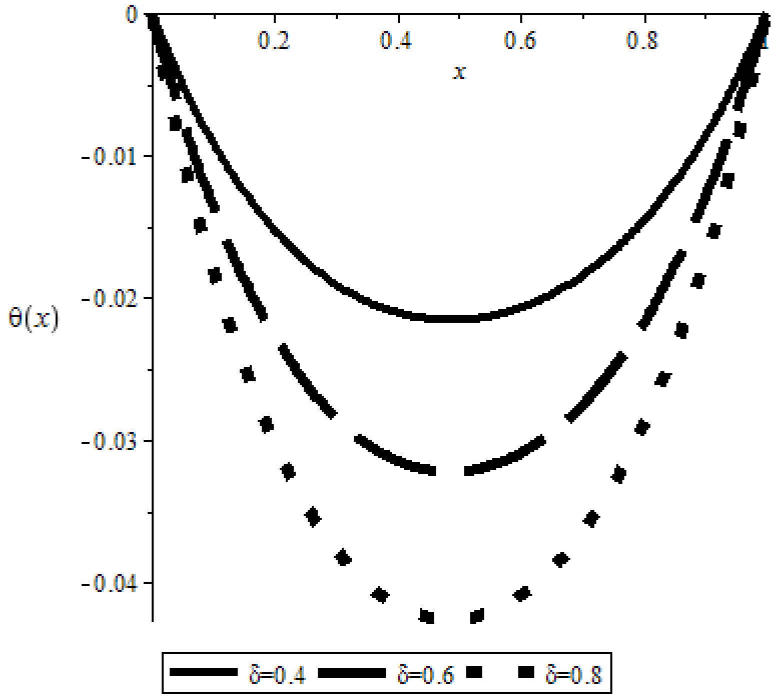

Figure 2: Plots of $\theta(x)$ against x for equation (11) for different values of δ and $\alpha = 12, \beta = 0.2, k_1 = 1, k_2 = 1, \epsilon = 0.01, e = 2.718$

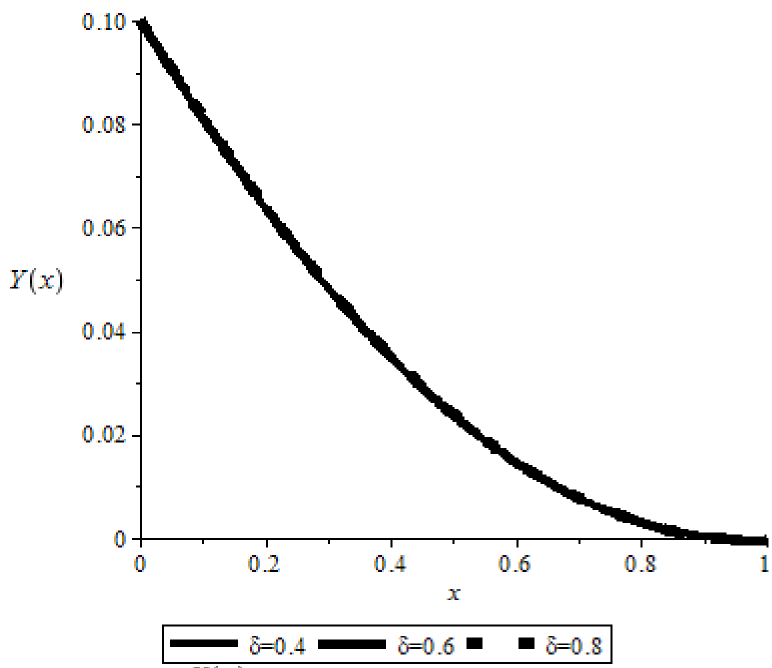

Figure 3: Plots of $Y(x)$ against x for equation (11) and (12) for different values of δ and $\alpha = 12, \beta = 0.2, k_1 = 1, k_2 = 1, \epsilon = 0.01, e = 2.718$

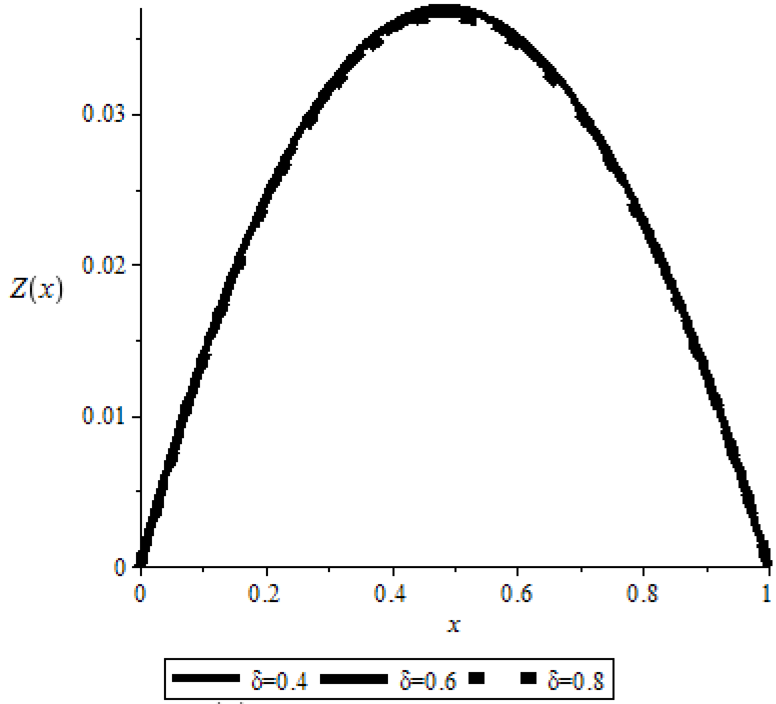

Figure 4: Plots of $Z(x)$ against x for equation (11) and (13) for different values of δ and $\alpha = 12, \beta = 0.2, k_1 = 1, k_2 = 1, \epsilon = 0.01, e = 2.718$

It is worth pointing out that the effect of δ as shown in Figures 2 and 3 physically means that the temperature is decreased and species is consumed in the spatial direction. These occur as a result of the oxidizer flux to and the heat losses from the reaction zone.

5.0 Conclusion

For the slow burning process associated with porous solid, analytical solution via asymptotic expansions is obtained for steady-state. The governing parameter for the problem under study is the Frank-Kamenetskii number. The analytical method is used to search for steady state temperature and species mass fraction profiles. The temperature and species mass fraction profiles are significantly influenced by the parameter involved. The analytical solution of the problem may help model numerical solutions and codes. It may be used as a preliminary predictive tool to study mathematically the slow burning process associated with porous solid. The work may be extended to more complex cases such as transient state and two-dimensional cases and therefore, recommended for further research.

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