Journal of the Nigerian Association of Mathematical Physics
***Volume* 23 (March, 2013), pp 197 – 202**
© J. of NAMP

The Application of Differential Transform Method to Solution of Linear Eight-Order Ordinary Differential Equation With Boundary Conditions.

Modebei Mark Ifeanyi and Awosusi Bola Monisola
Department of Mathematics,
Faculty of Sciences, University of Ibadan, Nigeria

Abstract

In hydrodynamic and hydro magnetic it is well known that when a layer of fluid is heated from below and is subject to the action of rotation, instability may set in as over stability. This instability may be modeled by an eighth-order ordinary differential equation with appropriate boundary conditions.The differential transform method is applied to construct numerical analytic solutions of linear eight-order ordinary differential equation with two-point boundary value conditions. The difference (errors) between the series solution obtained and the exact solution is also estimated and plot of graphs showing the effectiveness of this method.

Keywords: Differential transform method, Boundary value problem System of equations, Taylor series.

1.0 Introduction

In this paper, we consider the following eight-order boundary value problem of the form;

$$y^{(viii)} + f(x)\,y(x) = g(x) \qquad a < x < b \qquad \textbf{(1.1)}$$

$$y(a) = \alpha_0 \quad y(b) = \alpha_1 \quad y''(a) = \varepsilon_0 \quad y''(b) = \varepsilon_1$$

$$y^{(iv)}(a) = \beta_0 \quad y^{(iv)}(b) = \beta_1 \quad y^{(vi)}(a) = \sigma_0 \quad y^{(vi)}(b) = \sigma_1 \qquad \textbf{(1.2)}$$

Where $\alpha_i,\ \varepsilon_i,\ \beta_i$ and $\sigma_i\ \ i = 0,1$ are real constants while the function $f(x)$ and $g(x)$ are continuous function on the closed and bounded interval $[a, b]$.

Eight-order ordinary differential equation arises in application of physical sciences especially in fields of engineering. A class of characteristic-value problems of higher order (as higher as twenty four) is known to arise in hydrodynamic and hydro magnetic stability [1]. In addition, it is well known that when a layer of fluid is heated from below and is subject to the action of rotation, instability may set in as over stability [1, 2, 3]. This instability may be modeled by an eighth-order ordinary differential equation with appropriate boundary conditions [1, 4].

In this paper, we shall find the solution of the above stated boundary value problem using the differential transform method.

Vedat [5], studied differential transform method to linear sixth-order BVP in 2007. Agarwal [6] also studied boundary value problems for higher order differential equation in 2007. Boutayeb and Twizell [7], employed Finite difference method to find the solution of eighth order boundary value problems. The results obtained were divergent at points adjacent to the boundary.

The differential transform method was also employed by Arıko˘glu and Ozkol [8], to find the solution of difference equation.

The method that we used is the differential transform methods (DTM), is based on Taylor series expansion. It was introduced by Zhou [9] in a study about electric circuits. It gives exact value of the differential transform of the Kth derivatives of an analytic function at a point in terms of known and unknown boundary conditions in a fast manner.

2.0 Differential Transform Method

Definition.

The differential transform of the Kth derivative of the function $f(x)$ is defined as follows:

$$F(K) = \frac{1}{K!}\left[\frac{d^K f(x)}{dx^K}\right]_{x=x_0} \qquad (\mathbf{2.1})$$

and the differential inverse transform of $F(K)$ is defined as follows:

$$f(x) = \sum_{K=0}^{\infty} F(K)(x - x_0)^K \qquad (\mathbf{2.2})$$

Corresponding author ***Modebei Mark,*** E-mail: gmarc.ify@gmail.com-, Tel. +2348051749774

In real application, the function $f(x)$ is expressed as a finite series and (2.2) can be written as

$$f(x) = \sum_{K=0}^{n} F(K)(x - x_0)^K \qquad (\mathbf{2.3})$$

As a result of the equations (2.1) and (2.2), defined above, some important theorems can be deduced and these theorems are used in line to obtain some basic result and computation required in illustrating this method.
Equations (2.1) and (2.2) shows that this method uses the Taylor's expansion .
i.e.

$$f(x) = \sum_{K=0}^{\infty} (x - x_0)^K \frac{1}{K!} \left[\frac{d^K f(x)}{dx^K}\right]_{x=x_0} \qquad (\mathbf{2.4})$$

3.0 Differential Transform of Some Common Functions.

The table 3.1 shows the differential transform of some common functions.
Table 3.1 Functions and their differential Transform

Original functions	Transforme function
$Sin(x)$	$S(K) = \begin{cases} \frac{(-1)^{\frac{K-1}{2}}}{K!} & if\ K\ is\ odd \\ 0, & if\ K\ is\ even \end{cases}$
$Cos(x)$	$C(K) = \begin{cases} \frac{(-1)^{\frac{K}{2}}}{K!} & if\ K\ is\ \text{even} \\ 0, & if\ K\ is\ \text{odd} \end{cases}$
$f(x) = e^x$	$F(K) = \frac{1}{K!}$
$f(x) = \lambda x^n$	$F(K) = \lambda\delta(K + n), where \quad \delta(K + n) = \begin{cases} 1 & if\ K = n \\ 0 & if\ K \neq n \end{cases}$ λ is a constant.
$Sin(nx)$	$S_n(K) = \begin{cases} \frac{\mathrm{n}^{\mathrm{K}}(-1)^{\frac{K-1}{2}}}{K!} & if\ K\ is\ odd \\ 0, & if\ K\ is\ even \end{cases}$
$Cos(nx)$	$C_{\mathrm{n}}(K) = \begin{cases} \frac{\mathrm{n}^{\mathrm{K}}(-1)^{\frac{K}{2}}}{K!} & if\ K\ is\ \text{even} \\ 0, & if\ K\ is\ \text{odd} \end{cases}$
$f_n(x) = e^{nx}$	$F_n(K) = \frac{n^K}{K!}$
$f(x) = \frac{d\,u(x)}{dx}$	$F(K) = (K + 1)U(K + 1)$

4.0 Basic Theorems

Theorem 4.1

Suppose the differential transform of the Kth derivative of the functions $f(x), g(x)$, and $h(x)$ are respectively,

$$F(K) = \frac{1}{K!}\left[\frac{d^K f(x)}{dx^K}\right]_{x=x_0}, \quad G(K) = \frac{1}{K!}\left[\frac{d^K g(x)}{dx^K}\right]_{x=x_0}, \quad H(K) = \frac{1}{K!}\left[\frac{d^K h(x)}{dx^K}\right]_{x=x_0}$$

If $f(x) = g(x) \pm h(x)$, then the differential transform of the Kth derivative of the function $f(x)$ is given by $F(k) = G(k) \pm H(k)$

Theorem 4.2

If $f(x) = cg(x)$, and the differential transform of the Kth derivative of the function $g(x)$ is $G(K) = \frac{1}{K!}\left[\frac{d^K g(x)}{dx^K}\right]_{x=x_0}$, then the differential transform of the Kth derivative of the function $f(x)$ is $F(k) = c\ G(k)$, where c is a constant.

Theorem 4.3

$If\ f(x) = \frac{d^n\, g(x)}{dx^n}$, and the differential transform of the Kth derivative of $g(x)$ is $G(K) = \frac{1}{K!}\left[\frac{d^K g(x)}{dx^K}\right]_{x=x_0}$, then the differential transform of the Kth derivative of $f(x)$ is given by

$$F(K) = \frac{(K+n)!}{K!}G(K+n).$$

Theorem 4.4
If $f(x) = g(x)h(x)$, then the differential transform of the Kth derivative of the function $f(x)$ is given by

$$F(K) = \sum_{k_1=0}^{K} G(K_1)H(K-K_1)$$

Theorem 4.5
If $f(x) = x^n$, then the differential transform of the Kth derivative of $f(x)$ is given by

$$F(K) = \delta(K+n)\,, where \quad \delta(K+n) = \begin{cases} 1 & if\ K = n \\ 0 & if\ K \neq n \end{cases}$$

Theorem 4.6
If $f(x) = g_1(x)g_2(x)\cdots\cdots\cdots g_{n-1}(x)g_n(x)$ then we define the differential transform of the Kth derivative of the function $f(x)$ by

$$F(K) = \sum_{K_{n-1}=0}^{K}\sum_{K_{n-2}=0}^{K_{n-1}}\cdots\cdots\sum_{K_2=0}^{K}\sum_{K_1=0}^{K_2} G_2(K_1)G_1(K_2-K_1)\times\cdots\cdots\times G_{n-1}(K_1-K_2)G_n(K-K)$$

5.0 Numerical Results

The numerical result using the differential transform method (DTM) is then applied to the linear eight-order boundary value problem of the form of (1.1) with the boundary conditions in (1.2)

Example.
Consider the following linear boundary value problem of eight-order

$$y^{(viii)}(x) = -8e^x + y(x), \qquad 0 < x < 1$$

Subject to the boundary conditions

$$y(0) = 1,\ \ y''(0) = -1,\ \ y^{(lv)}(0) = -3,\ \ y^{(vi)}(0) = -5,$$
$$y(1) = 0,\ \ y''(1) = -2e,\ \ y^{(lv)}(1) = -4e,\ \ y^{(vl)}(1) = -6e$$

Solution
Now, by transforming the equation step-wise we obtain the following:
By transforming $y^{(viii)}(x)$ using theorem (4.3) we have;

$$\frac{(K+8)!}{K!}Y(K+8);$$

By transforming $-8e^x$, from the table of transformed functions, we have;

$$-\frac{8}{K!};$$

and finally by transforming $y(x)$, by definition, we have;

$$Y(K).$$

Hence the total transform of the given equation becomes;

$$\frac{(K+8)!}{K!}Y(K+8) = -\frac{8}{K!} + Y(K)$$

By simplification we obtain;

$$Y(K+8) = \frac{1}{(K+1)(K+2)\times\cdots\times(K+8)}\left(-\frac{8}{K!} + Y(K)\right) \qquad (\mathbf{5.0})$$

This is the recurrence relations, where we obtain the value of

$$Y(8),\quad Y(9),\quad Y(10),\ldots \quad for\ K = 0,1,2,3,\ldots.$$

The differential transform of the boundary conditions of the given differential equation at $x_0 = 0$ are:

$$\begin{cases} Y(0) = 1, \quad Y(2) = -\frac{1}{2}, \quad Y(4) = -\frac{1}{8}, \quad Y(6) = -\frac{1}{144}, \quad \sum_{K=0}^{n} Y(K) = 0, \\ \sum_{K=0}^{n} K(K-1)Y(K) = -2e, \quad \sum_{K=0}^{n} K(K-1)(K-2)(K-3)Y(K) = -4e \qquad (\mathbf{5.1}) \\ \sum_{K=0}^{n} K(K-1)(K-2)(K-3)(K-4)(K-5)Y(K) = -6e \end{cases}$$

Using the transformed boundary conditions in (5.1) we obtain the series solution as follows

$$y(x) = 1 + a_1 x - \frac{1}{2}x^2 + a_3 x^3 - \frac{1}{8}x^4 + a_5 x^5 - \frac{1}{144}x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9 + a_{10}x^{10} + a_{11}x^{11} + a_{12}x^{12} + Ox^{13} \qquad (\mathbf{5.2})$$

Since $a_n = Y(n) = \frac{y^n(0)}{n!}$, we use the recursive relation of (5.0) to obtain the following

$$a_8 = Y(8) = -\frac{1}{5760};\ a_9 = Y(9) = \left[-\frac{1}{45360} + \frac{a_1}{362880}\right];\ a_{10} = Y(10) = -\frac{1}{403200}$$

$$a_{11} = Y(11) = \left[-\frac{1}{4989600} + \frac{a_3}{39916800}\right];\ a_{12} = Y(12) = -\frac{1}{43545600} \qquad (\mathbf{5.3})$$

$for\ k = 0, 1, 3, 4$, respectively.

Using the transformed boundary conditions in (5.1) for $n = 12$, we get

$$Y(0) + Y(1) + Y(2) + Y(3) + Y(4) + Y(5) + Y(6) + Y(7) + Y(8) + Y(9) + Y(10) + Y(11) + Y(12) = 0$$

$$2Y(2) + 6Y(3) + 12Y(4) + 20Y(5) + 30Y(6) + 42Y(7) + 56Y(8) + 72Y(9) + 90Y(10) + 110Y(11) + 132Y(12) = -2e$$

$$24Y(4) + 120Y(5) + 360Y(6) + 840Y(7) + 1680Y(8) + 3024Y(9) + 5040Y(10) + 7920Y(11) + 11880Y(12) = -4e$$

$$720Y(6) + 5040Y(7) + 20160Y(8) + 60480Y(9) + 151200Y(10) + 332640Y(11) + 665280Y(12) = -6e$$

By substitution of (5.3) and other known values we have

$$1 + a_1 - \frac{1}{2} + a_3 - \frac{1}{8} + a_5 - \frac{1}{144} + a_7 - \frac{1}{5760} + \left[-\frac{1}{45360} + \frac{a_1}{362880}\right] - \frac{1}{403200} + \left[-\frac{1}{4989600} + \frac{a_3}{39916800}\right] - \frac{1}{43545600} = 0$$

$$-1 + 6a_3 - \frac{3}{2} + 20a_5 - \frac{1}{48} + 42a_7 - \frac{56}{5760} + \left[-\frac{72}{45360} + \frac{72a_1}{362880}\right] - \frac{90}{403200} + \left[-\frac{110}{4989600} + \frac{110a_3}{39916800}\right] - \frac{132}{43545600} = -2e$$

$$-3 + 120a_5 + -\frac{360}{144} + 840a_7 - \frac{1680}{5760} + \left[-\frac{3024}{45360} + \frac{3024a_1}{362880}\right] - \frac{5040}{403200} + \left[-\frac{7920}{4989600} + \frac{7920a_3}{39916800}\right] + -\frac{11880}{43545600} = -4e$$

$$-\frac{720}{144} + 5040a_7 - \frac{20160}{5760} + \left[-\frac{60480}{45360} + \frac{60480a_1}{362880}\right] - \frac{151200}{403200} + \left[-\frac{332640}{4989600} + \frac{332640a_3}{39916800}\right] - \frac{665280}{43545600} = -6e$$

So that by simplifying we obtain the following system of equations

$$\frac{362881}{362880}a_1 + \frac{39916801}{39916800}a_3 + a_5 + a_7 = -0.367857195$$

$$\frac{1}{5040}a_1 + \frac{329500910}{39916800}a_3 + 20a_5 + 42a_7 = -2.716672508$$

$$\frac{3024}{362880}a_1 + \frac{7920}{39916800}a_3 + 120a_5 + 840a_7 = -4.966331678$$

$$\frac{60480}{362880}a_1 + \frac{332640}{39916800}a_3 + 5040a_7 = -6.019413191$$

Then by solving for $a_{1,}, a_3, a_5, a_7$ we obtain the following values

$$a_{1,} = -6.771 \times 10^{-7},\quad a_3 = -0.3333344127,\quad a_5 = -0.033332858585,\quad a_7 = -0.00119054874 \qquad (\mathbf{5.4})$$

So by substituting the values in (5.3) and (5.4) into (5.2) we have

$$y(x) = 1 - 6.771 \times 10^{-7}x - \frac{1}{2}x^2 - 0.3333344127x^3 - \frac{1}{8}x^4 - 0.033332858585x^5 - \frac{1}{144}x^6 - 0.00119054874x^7 - \frac{1}{5760}x^8 - 2.20458 \times 10^{-5}x^9 - \frac{1}{403200}x^{10} - 2.50503 \times 10^{-7}x^{11} - \frac{1}{43545600}x^{12} + Ox^{13}$$

This solution obtained can be compared with the closed exact solution of the BVP given which is $y(x) = (1 - x)e^x$. The table below gives the comparison.

Table 5.1 Differences between series solution obtained and exact solution

x	Series solution	Exact solution	Error
0.0	1	1	0.0000000
0.2	0.9771221	0.9771222	1.4390394×10^{-7}
0.4	0.8950945	0.8950948	3.3517633×10^{-7}
0.6	0.7288469	0.7288475	6.0443279×10^{-7}
0.8	0.4451072	0.4451082	9.516282×10^{-7}
1.0	0.0000000	0.0000000	0.0000000

$Error = Exact\ Solution - Series\ Solution$

Table 5.1 exhibits the numerical values for series solution and that of the exact solution along with the errors obtained. We observed that the errors are minimal. By using the DTM, It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of $y(x)$.

Figure 5.1 (a) - Series Solution

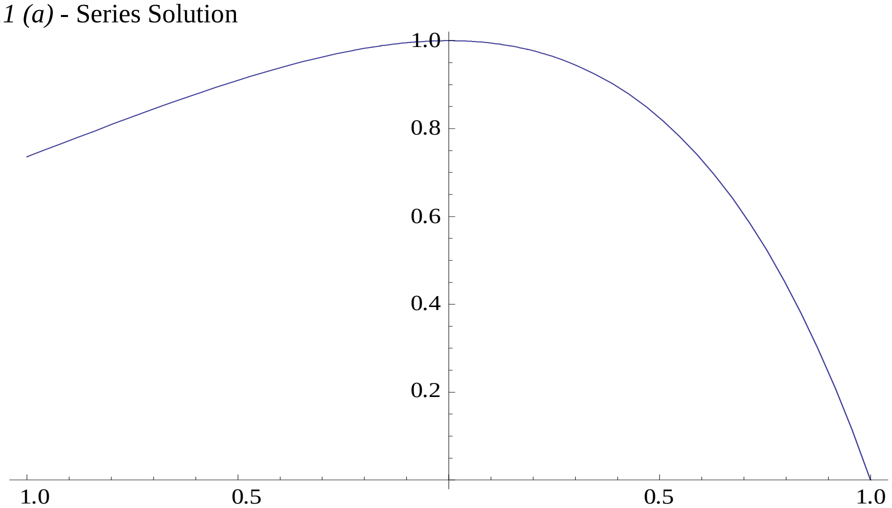

Figure 5.1 (b) - Exact Solution

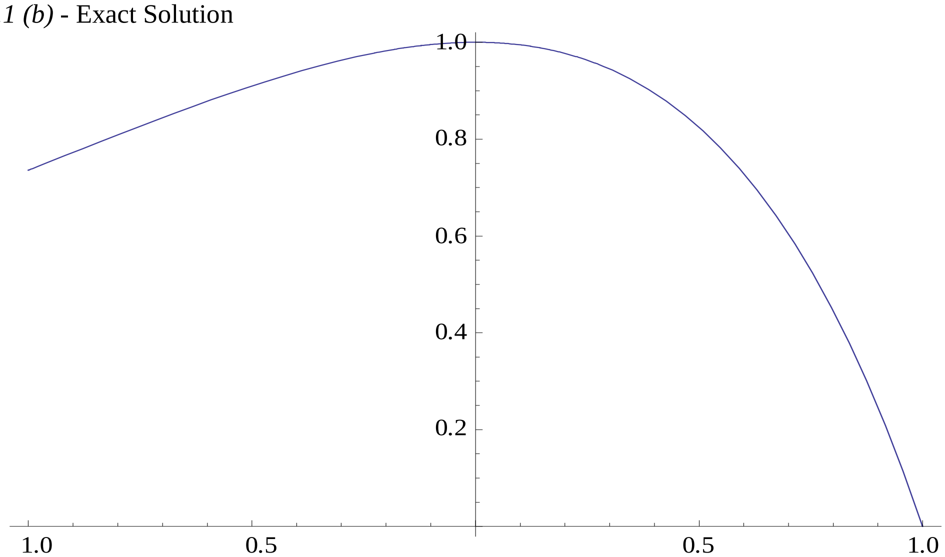

Figure 5.1 (a) and Figure 5.1 (b) shows the graphs of the series solution and the exact solution. We see that both series solution obtained by the differential transform method and the exact solution are almost the same with nearly insignificant difference (error).

6.0 Conclusion

In this paper, we studied differential transform method for solving linear ordinary differential equation of eight-order with boundary conditions. This method was applied to solve some boundary value problems. In the Example, we obtained the exact series solution for the BVP. It is observed that the method is an effective and reliable tool for the solution of such problems.

It may be concluded that Differential Transform Method (DTM) is very powerful and efficient in finding the analytic solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. Thus we conclude that differential transform method can be considered as an efficient method for solving linear BVP.

References

[1] Chandrasekhar S.- Hydrodynamic and Hydro-magnetic Stability, Clarendon Press, Oxford (Reprinted: Dover Books, New York, , 1961).

[2]. NoorM.A. and Mohyd-Din S.T.- Homotopy perturbation method for solving eighth order boundary value problem J. Math. Anal. App. Thy. 1 (2) (2006) 161-169.

[3].Siddiqi S.S and Twizell E.H.- Spline solution of linear eighth-order boundary value problems, Comput. Math. Appl. Mech. Engrg. 131(1996) 309-325.

[4]. Wazwaz, A. M.- The numerical solutions of special eighth-order boundary value problems by the modified decomposition method, Neu. Parl. Sci. Comp. 8(2) (2000) 133-146.

[5]Vedat Suat Ert¨urk- Application of differential transform method to linear sixth- order boundary value problems Applied Mathematical Sciences, Vol. 1, (2007) no. 2, 51 – 58

[6]Agarwal R.P. - Boundary Value Problems for Higher-Order Differential Equations, World Scientific, Singapore (1986).

[7]Boutayeb and E. H. Twizell- Finite-difference methods for the solution of special eighth-order boundary-value problems, Int. J. Comput. Math., 48, (1993) 63-75.

[8] Arıkoˇglu, A. Ozkol, I- Solution of difference equations by using differential transform method, Appl. Math. Comput. 174(2006) 1216-1228.

[9] Zhou, J. K. Differential Transform and Its Applications for Electrical Circuits (in Chinese). Huazhong University Press, Wuhan, China. (1986).