Journal of the Nigerian Association of Mathematical Physics
Volume **23 (March, 2013), pp 193 – 196**
© J. of NAMP

An Analysis of a One-Dimensional, Non Reactive Contaminant flow with Variable Transport Coefficients.

Aiyesimi , Y. M. and Iyeme, E. E.

Department of Mathematics/Statistics
Federal University of Technology, Minna.

Abstract

In this work, a one dimensional, non-reactive advection-dispersion equation model with variable coefficients which are dependent on the position variable is presented. The model is solved using the Homotopy perturbation method. The analytical solution is analyzed with the help of a set of input data to understand the behaviour of the concentration distribution. The solute concentration distribution as revealed in the graphs agree with the linearly time dependent assumption in real life.

Keywords: Non-reactive, Advection, Dispersion, Variable coefficients

1.0 Introduction

Analysis and predictions of solute transport (contaminants) in hydro geological systems generally involve the use of some form of partial differential equations of parabolic type usually referred to as Advection and Dispersion Equation. Although the usefulness of the mass balance equation on which the Advection and Dispersion Equation (ADE) is based to describe transport in natural soils due to the variability of flow and transport properties in the field is still under scientific scrutiny, it is likely, that Advection and Dispersion equation will remain relevant as a potent tool for research purposes thus it will be used in this work.

Most works on Advection and Dispersion equation studies of late, presented the advection and dispersion coefficients as constants determined by laboratory experiments. However Aral and Liao[1] examined solutions to two dimensional ADE, with time –dependent dispersion. Sirin[2] assumed pore flow velocity to be a non – divergence free, unsteady and non stationary random function of space and time.

Yates[3] developed an analytical solution' for describing transport of dissolved substances in heterogeneous porous media with a distance dependent dispersion. Van Kootein[4] gave a method to solve the transport equation for a kinetically adsorbing solute in a porous medium with spatially varying velocity and dispersion coefficients. Chen et al.[5] described solute transport in a radially convergent flow field with scale- dependent dispersion. Other contributors to the studies of ADE with variable coefficients includes, Meerschaert and Tadjeran[6], Zoppou and Knight[7] and Jaiswal and Kumar[8].

In this work, the Advection and Dispersion coefficients of a non reactive ADE in one dimension are both considered as functions of distance. Due to the non-linear nature of the resulting ADE, a semi analytical method will give a better approximation of the solution where such solution exists as against the analytical methods applied in previous works. Hence semi analytical solution of these non reactive ADE in one dimension describing the dispersion of a continuous type, input point source discharge in a moving stream, through semi-infinite medium is presented using the He's Homotopy Perturbation Method.

2.0 Mathematical Formulation of the Problem:

We consider a general ADE in one-dimension as follows:

$$\frac{\partial C}{\partial t} = D_L \frac{\partial^2 C}{\partial x^2} - U_L \frac{\partial C}{\partial x} \tag{1}$$

Corresponding author: ***Iyeme E. E.,*** E-mail: emengemeng@yahoo.com, Tel.: +2348137953085

where,

$C(x,t)$ is the concentration at any time t, in a horizontal plane, $D_L[\frac{L^2}{T}]$ is the coefficient of Dispersion, $U_L[\frac{L}{T}]$ is the coefficient of Advection, x [L] is the distance and t [T] is time.

Let $U_L = U(x) = sx$, (3)

and $D_L = D(x) = kx^2$ (4)

where s and k are constants of dimension, $[\frac{1}{T}]$.

Thus equations (1) and (2) is transformed into:

$$\frac{\partial C}{\partial t} = (kx^2)\frac{\partial^2 C}{\partial x^2} - (sx)\frac{\partial C}{\partial x} \quad (5)$$

An analytical solution of equation (5) using the Homotopy Perturbation method is sought under the following initial and boundary conditions:

$$C(x,0) = e^{-\lambda x}, \; C_t(x,0) = 0 \text{ and } C(\infty,t) = 0, \; \lambda > 0 \quad (6)$$

$\lambda[\frac{1}{T}]$ is the flow resistance coefficient. In order to solve equation (5) we employ He's Homotopy Perturbation method (HPM) which is constructed as follows:

$$(1-p)[\frac{\partial V}{\partial t}] + p[\frac{\partial V}{\partial t} - (kx^2)\frac{\partial^2 V}{\partial x^2} + (sx)\frac{\partial V}{\partial x}] = 0 \quad (7)$$

It is assumed that the solution of equation (5) can be written as power series of p as follows:

$$V = V^{(0)} + pV^{(1)} + p^2V^{(2)} + \dots \quad (8)$$

And the best approximation for the solution of equation (5) is:

$$C = \lim_{p\to 1} V = V^{(0)} + V^{(1)} + V^{(2)} + \dots \quad (9)$$

See He, J.[9-10],Ghasemi et al.[11].

Substituting equation (8) into (7) and equating the equal powers of p terms one has:

$$p^0 : \frac{\partial V^{(0)}}{\partial t} = 0 \quad (10)$$

$$p^1 : \frac{\partial V^{(1)}}{\partial t} = \frac{\partial V^{(0)}}{\partial t} - (kx^2)\frac{\partial^2 V^{(0)}}{\partial x^2} + (sx)\frac{\partial V^{(0)}}{\partial x} \quad (11)$$

$$p^2 : \frac{\partial V^{(2)}}{\partial t} = \frac{\partial V^{(1)}}{\partial t} - (kx^2)\frac{\partial^2 V^{(1)}}{\partial x^2} + (sx)\frac{\partial V^{(1)}}{\partial x} \quad (12)$$

The solution of equations (10) , (11) and (12) yields :

$$V^{(0)} = e^{-\lambda x} \quad (13)$$

$$V^{(1)} = \lambda x e^{-\lambda x}(k\lambda x + s)t + f_1(x) \quad (14)$$

$$V^{(2)} = \frac{1}{2}\lambda x e^{-\lambda x}(k^2x^3\lambda^3 + 2kx^2\lambda^2 s - 4k^2x^2\lambda^2 - 4\lambda xk^2 + 2\lambda xk^2 + s^2x\lambda - s^2)t^2 + f_2(x). \quad (15)$$

The leading equation inherits the auxiliary conditions of the initial problem while all higher terms satisfies homogenous auxiliary condition.

From equation (8) and (9), the solution of equation (5) becomes:

$$C(x,t) = e^{-\lambda x} + \lambda x e^{-\lambda x}(k\lambda x + s)t + \frac{1}{2}\lambda x e^{-\lambda x}(k^2x^3\lambda^3 + 2kx^2\lambda^2 s - 4k^2x^2\lambda^2 - 4\lambda xks$$
$$+2\lambda xk^2 + s^2x\lambda - s^2)t^2 + O(p^3). \quad (16)$$

This is the solution of a non reactive contaminant flow with variable transport coefficients.

3.0 Result and discussions

The analytical solutions are illustrated with the help of a set of input data to understand the behavior of the concentration distribution. The different variables are assigned numerical values, as follows; λ =1.0 (1/day),

k =0.01(1/day), s =0.1 (1/day). The dispersion coefficient is considered as the square of the advection coefficient. All the figures are drawn with the help of the Maple software which also aided in the solutions of the ordinary differential equations which resulted from the Homotopy Perturbation Method.

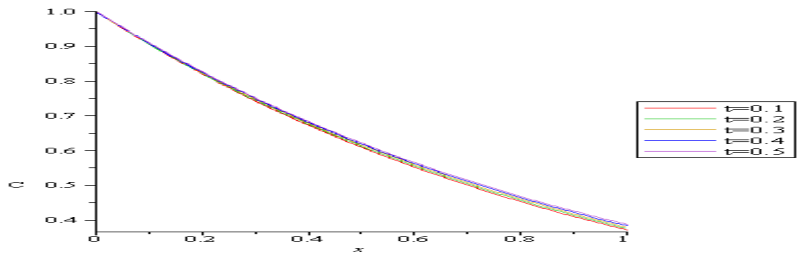

Fig.1.Behavior of concentration of non reactive, one-dimensional, ADE at time t=0.1 to 0.5.

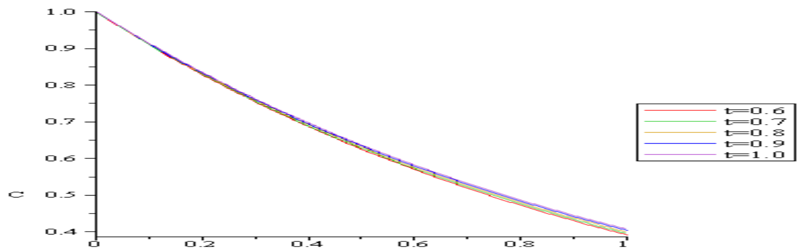

Fig.2.Behavior of concentration of non reactive, one-dimensional, ADE at time t=0.6 to 1.0.

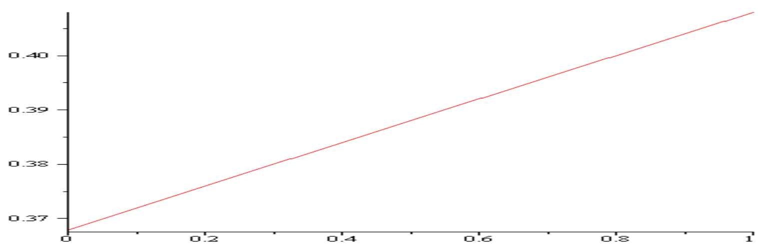

Fig.3 Behaviour of concentration of non reactive, one-dimensional ADE, against time $(x=1, s=0.1, k=0.01, \lambda=1)$.

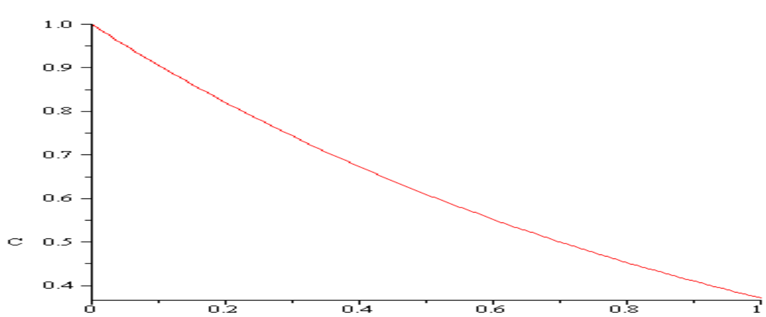

Fig.4. Behaviour of concentration of non reactive, one-dimensional ADE, at different flow resistance coefficient λ $(x=1, s=0.1, k=0.01, t=1)$.

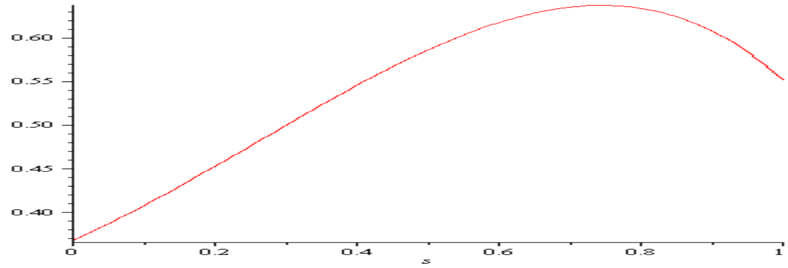

Fig.5. Behaviour of concentration of non reactive, one-dimensional ADE, against the velocity component s $(x=1, k=s^2, t=1, \lambda=1)$.

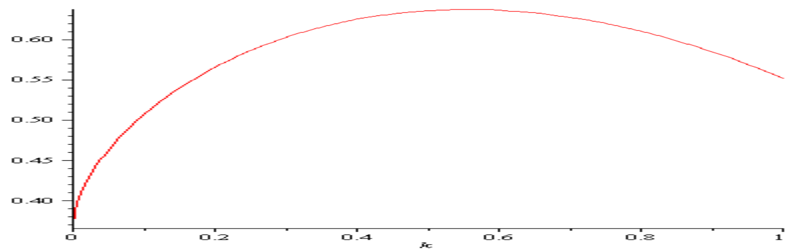

Fig.6. Behaviour of concentration of non reactive, one-dimensional ADE, against the dispersion component k $(x=1, s=k^{0.5}, t=1, \lambda=1)$.

4.0 Conclusions:

The governing solute transport equationwith variable coefficients dependent on the position variable is solved using He's Homotopy Perturbation Method. The proposed solution can be applied to field problems where the hydrological properties of the medium and prevailing boundary and initial conditions are the same or an approximation of the ones considered in this study. The behavior of the solute concentration as revealed in the graphs agree with the linearly time dependent assumption in real life.

References

[1]. Aral, M.M.;and Liao, B., "Analytical solutions for two-dimensional transport equation with time dependent dispersion coefficients", *Journal of Hydrologic Engineering,1(1),pp. 20-32 (1996).*

[2]. Sirin, H., "Ground water contaminant transport by non divergence-free, unsteady and non-stationary velocity fields", *Journal of Hydrology, Vol. 330, pp. 564-572(2006).*

[3]. Yates, S.R., "An analytical solution for one dimensional transport in heterogeneous porous media", *Water Resources Research, Vol. 26, pp.2331-2338(1990).*

[4]. Van Kooten, J.J.A., "A method to solve the advection-dispersion equation with kinetic adsorption isotherm", *Advances in Water Resources, Vol. 19, pp. 193-206(1996).*

[5]. Chen, J. S.,Liu, C,W., Hsu, H.T. and Liao, C.M., "A Laplace transforms power series solution for solute transport in a convergent flow field with scale-dependent dispersion", *Water Resources Research, Vol. 39, No. 8, pp.1229(2003).*

[6]. Meerschaert, Mark M. and Tadjeran, Charles, "Finite difference approximations for fractional advection-dispersion flow equations", *Journal of Computational and Applied Mathematics, Vol.172, pp. 65-77(2004).*

[7]. Zoppou, C., and Knight, J.H., "Analytical solutions for advection and advection-diffusion equation wit spatially variable coefficients", *Journal of Hydraulic Engineering, Vol.123, pp. 144-148(1997).*

[8]. Jaiswal, D.K., Kumar A., "Analytical solutions of advection-dispersion equation for varying pulse type input point source in one-dimension", *International Journal of Engineering, Science and Technology, Vol.3, pp. 22-29(2011).*

[9]. He J., "An approximate solution technique depending on an artificial parameter", *Communications in Nonlinear science and Numerical Simulation, Vol.3,pp. 92-97(1998).*

[10]. He J., "A coupling method of a Homotopy technique and perturbation technique for nonlinear problems", *International journal of nonlinear mechanics,Vol.35 pp. 37-43(1998).*

[11]. Ghasemi, M.,Babolian, E. ,Saeidiain, J. and Azizi, A., "A general approach for Homotopy Perturbation method for differential and systems of differential equations", *Australian Journal of Basic and Applied Sciences, 5(12), pp. 3280-3294(2011).*