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A Matlab Programming Approach for the Sensitivity of Model Parameters of a Prey-Predator Interaction with Harvesting

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Abstract

Following the estimated parameterization of Khamis et al. [2011] such that $N_1(0) = 50$, $N_2(0) = 50$, $P(0) = 45$, carrying capacities $K_1 = 110$ and $K_2 = 100$, the migration rate of prey species in the free zone and the migration rate of prey species in the reserve zone have precise values of 0.5 and 0.4 respectively. We considered the instance when the duration of interaction is 180 days. On the basis of sensitivity analysis, we have found that the migration rate of prey species in the free zone is dominantly a more sensitive parameter than the migration rate of prey species in the reserve zone when the 1-norm ODE 45 and 2-norm ODE 45 sensitivity schemes were implemented on our tested Matlab program codes. On the other hand, a similar infinity-norm ODE 45 sensitivity measures show that the migration rate of prey species in the free zone and the migration rate of prey species in the reserve zone can be classified as equally sensitive parameters or equally important parameters of this aquatic ecosystem. Using a 1-norm ODE 45 when both migration rates are varied by 1 per cent, the cumulative percentage changes in the solution trajectories or model outputs are 147.37 and 89.17. Other numerical results of sensitivity measures are presented and discussed.

Keywords: Matlab Programming Approach, Sensitivity, Prey-Predator Interaction, Harvesting

1.0 Introduction

The prey-predator interaction with harvesting is an important interdisciplinary active research application area which has recently been studied by a team of expert African-European mathematicians [1]. A model for two fish species and one predator in a patchy environment was formulated using a deterministic model to study the dynamics of fishery in two homogeneous patches, a free fishing zone and a refuge for prey reserve in which fishing was prohibited. The system of three model equations of continuous nonlinear first order ordinary differential equations were analysed around steady states while the criteria for local and global stabilities were also established. Next, the existence of bionomic equilibrium of the system was determined and the conditions for their existence were derived.

A particular method of sensitivity analysis was used to measure the relative change in state variables when parameters change [1]. Their sensitivity method is quite different both in principle, philosophy and theory from our present standard method of sensitivity which studies the sensitivity analysis of model parameters over a time interval otherwise called parametric sensitivity which is mathematically tractable with a system of continuous nonlinear first order ordinary differential equations of the Lotka-Volterra type which evolve over time. For the detailed explanation of this method, see one of our papers which have been published in the present volume of this journal.

Materials and Methods

The main source of data for this pioneering analysis is based on the data provided by Khamis et al. [1]. Their model is a system of continuous first order nonlinear ordinary differential equations with the following mathematical structure:

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$dN_1(t)/dt = r_1 N_1(t)(1-N_1(t)/K_1) + \beta_2 N_2(t) - \beta_1 N_1(t) - m_1 N_1(t) P(t) - qEN_1(t)$ (1)

$dN_2(t)/dt = r_2 N_2(t)(1-N_2(t)/K_2) - \beta_2 N_2(t) + \beta_1 N_1(t) - m_2 N_2(t)P(t)$ (2)

$dP(t)/dt = P(t)(-d - \sigma P(t) + \alpha_1 N_1(t) + \alpha_2 N_2(t))$ (3)

Here the initial conditions $N_1(0) > 0$, $N_2(0) > 0$ and $P(0) > 0$ and other model parameters are considered as positive constants. For the purpose of this simulation sensitivity analysis, the precise values of the migration rate of prey species in the free zone denoted by β_1 and the migration rate of prey species in the reserve zone denoted by β_2 are 0.5 and 0.4 respectively. The above model formulation describes a prey-predator interaction with harvesting in the context of aquatic ecosystem. Following Khamis et al. [1], we consider the prey in patch 1 denoted by $N_1(t)$ to be free for fishing and preys in patch 2 denoted by N_2 (t) as prey refuge which constitutes a reserve area and no fishing is permitted in that area. The predator population (density P(t)) has no barrier between the two patches in terms of fishing.

The method of sensitivity as proposed by Ekaka-a and Nafo [2] is as follows: vary a model parameter a little one-at-a-time and observe its impact on the solution trajectories. This variation will either produce a big effect on the solution trajectories or a small effect on the solution trajectories. The parameter which is associated with a big effect on the solution trajectories is called a sensitive parameter whereas the parameter which is associated with a small effect on the solution trajectories is called a least sensitive parameter. While a least sensitive parameter is considered as a rough estimate, the sensitive parameter should be estimated efficiently to minimise error in model predictions. These two outcomes of sensitivity characterizations are integral components of parameter estimation and model validation. In few scenarios, model parameters can be classified as both equally most sensitive and equally least sensitive.

The 1-norm calculation of the sensitivity of a model parameter is based on the theory of the sum of the data points of the difference between the solution trajectories when that parameter is not varied and the solution trajectories when that parameter is varied. The 2-norm calculation of the sensitivity of a model parameter is based on the theory of the square root of the data points of the difference between the solution trajectories when that parameter is not varied and the solution trajectories when that parameter is varied. The infinity-norm of the sensitivity of a model parameter is based on the theory of the maximum of the data points of the difference between the solution trajectories when that parameter is not varied and the solution trajectories when that parameter is varied. According to Ekaka-a [3], for each solution trajectory, the 1-norm value is expected to be bigger than its 2-norm value while the 2-norm value is expected to be bigger than its infinity-norm value. In summary, the value of the 1-norm is expected to be bigger than the values of the 2-norm and infinity-norm.

The core part of our algorithm which we have utilized to calculate the sensitivity of a model parameter is hereby described by the following steps:

- Identify and code the control system of model equations of continuous nonlinear first order ordinary differential equations in which the model parameter is not varied. For the purpose of this analysis, the three solution trajectories are denoted by N_1, N_2 and P ([3]).

- Identify and code a sub-model of the control system of model equations of continuous nonlinear first order ordinary differential equations in which the model parameter is varied one-at-a-time. In this scenario, the three solution trajectories are denoted by N_{1m}, N_{2m} and P_m ([3]).

- Code a Matlab program which runs the control model equations and the varied model equations ([3]).

- Under chosen initial conditions and a time range, the coded program will produce the solution trajectories for the control model equations and the varied model equations ([3]).

- Next, specify the difference between the solution trajectories of the control model equations and the varied model equations as $F_1 = N_1 - N_{1m}$, $F_2 = N_2 - N_{2m}$ and $P_3 = P - P_m$ ([3]).

- Calculate the 1-norm, 2-norm and infinity-norm for the three solution trajectories of the control model equations and similarly for the solution trajectories for the difference between the solution trajectories. For example, for the N_1 and N_{1m} solution trajectories which assume precise data points such as N_{1j} and N_{1jm} where the sub-script j takes on the values of 1, 2, 3, 4, 5, …n, the 1-norm for the N_1 solution trajectory is defined by the sum of the absolute values of N_{11}, N_{12}, N_{13}, N_{14}, N_{15}, up to the nth data point N_{1n}. In the same manner, the 2-norm for the N_1 solution trajectory is defined by the positive square-root of the sum of the squares of absolute values of N_{11}, N_{12}, N_{13}, N_{14}, N_{15}, up to the nth data point N_{1n}. The infinity-norm is defined by the maximum value of the set of the absolute values of N_{11}, N_{12}, N_{13}, N_{14}, N_{15}, up to the nth data point N_{1n}. The same procedure can be applied to calculate the 1-norm, 2-norm and infinity-norm of the N_2 and P solution trajectories ([3]).

- Having specified the difference between the solution trajectories of the control model equations and the varied model equations by $F_1 = N_1\text{-}N_{1m}$, $F_2 = N_2\text{-}N_{2m}$ and $F_3 = P\text{-}P_m$, for the given range of data points when j = 1, 2, 3, 4, 5, …n, the difference between the solution trajectories of the control model equations and the varied model equations will be $F_1 = N_{1j} – N_{1jm}$, $F_2 = N_{2j} – N_{2jm}$ and $F_3 = P_j – P_{jm}$.

- For the purpose of this analysis, we will also calculate the 1-norm, 2-norm and infinity-norm of F_1, F_2 and F_3. For example, the 1-norm of F_1 will be the sum of the absolute values of the data points $(N_{11} – N_{11m})$, $(N_{12} – N_{12m})$, $(N_{13} – N_{13m})$, $(N_{14} – N_{14m})$, $(N_{15} – N_{15m})$, … $(N_{1n} – N_{1nm})$ where N_{11} and N_{11m} stand for the first data point of the N_1 solution trajectory and the first data point of the modified N_{1m} solution trajectory, N_{12} and N_{12m} stand for the second data point of the N_1 solution trajectory and the second data point of the modified N_{1m} solution and so forth. The 1-norm of F_2 will be the sum of the absolute values of the data points $(N_{21} – N_{21m})$, $(N_{22} – N_{22m})$,
$(N_{23} – N_{23m})$, $(N_{24} – N_{24m})$, $(N_{25} – N_{25m})$, … $(N_{2n} – N_{2nm})$ where N_{21} and N_{21m} stand for the first data point of the N_2 solution trajectory and the first data point of the modified N_{2m} solution trajectory, N_{22} and N_{22m} stand for the second data point of the N_2 solution trajectory and the second data point of the modified N_{2m} solution and so forth. Similarly, the 1-norm of F_3 will be the sum of the absolute values of the data points $(P_1 – P_{1m})$, $(P_2 – P_{2m})$, $(P_3 – P_{3m})$, $(P_4 – P_{4m})$, $(P_5 – P_{5m})$, … $(P_n – P_{nm})$ where P_1 and P_{1m} stand for the first data point of the P solution trajectory and the first data point of the modified P_m solution trajectory, P_2 and P_{2m} stand for the second data point of the P solution trajectory and the second data point of the modified P_m solution and so forth. The 2-norm and infinity-norm can be similarly calculated for the differences of the three solution trajectories F_1, F_2 and F_3.

- To calculate the percentage variation in N_1 solution trajectory of a model parameter one-at-a-time when other parameters are fixed, we will calculate the following values: (1-norm of F_1 divided by the 1-norm of N_1) times 100; (2-norm of F_1 divided by the 2-norm of N_1) times 100; (infinity-norm of F_1 divided by the infinity-norm of N_1) times 100 ([3]).

- To calculate the percentage variation in N_2 solution trajectory of a model parameter one-at-a-time when other parameters are fixed, we will calculate the following values: (1-norm of F_2 divided by the 1-norm of N_2) times 100; (2-norm of F_2 divided by the 2-norm of N_2) times 100; (infinity-norm of F_2 divided by the infinity-norm of N_2) times 100 ([3]).

- To calculate the percentage variation in P solution trajectory of a model parameter one-at-a-time when other parameters are fixed, we will calculate the following values: (1-norm of F_3 divided by the 1-norm of P) times 100; (2-norm of F_3 divided by the 2-norm of P) times 100; (infinity-norm of F_3 divided by the infinity-norm of P) times 100 ([3]).

- Due to the unstable changing values of the 1-norm, 2-norm and infinity-norm specifications, we adopt to use a compact value for related percentage values of these norms. For example, the cumulative percentage value of 1-norm sensitivity in terms of the difference in solution trajectories involves the sum of the 1-norm value due to F_1 solution trajectory, 1-norm value due to F_2 solution trajectory and 1-norm value due to F_3 solution trajectory. The same procedure can be followed to calculate the cumulative percentage values of 2-norm and infinity-norm sensitivities in terms of the F_1 solution trajectory, F_2 solution trajectory and F_3 solution trajectory ([3]).

- The final test is to check if the 1-norm of the N_1 solution trajectory, N_2 solution trajectory and P solution trajectory satisfy the condition that the precise value of the 1-norm of N_1 solution trajectory (or N_2 solution trajectory or P solution trajectory) is bigger than the 2-norm of N_1 solution trajectory (or N_2 solution trajectory or P solution trajectory), followed by the condition that the precise value of 2-norm of N_1 solution trajectory (or N_2 solution trajectory or P solution trajectory) is bigger than the infinity-norm of N_1 solution trajectory(or N_2 solution trajectory or P solution trajectory) and the condition that the precise value of 1-norm of N_1 solution trajectory (or N_2 solution trajectory or P solution trajectory) is bigger than the infinity-norm of N_1 solution trajectory (or N_2 solution trajectory or P solution trajectory) ([3]).

What are we looking for? We want to find the migration parameter which when varied will have a big effect on the solution trajectory. On the basis of this computational technique, we can calculate the sensitivity of each migration parameter and differentiate their extent of sensitivity when the precise values of other model parameters are fixed.

Our computational method of calculating the sensitivity of the migration rate of prey species in the free zone and the migration rate of prey species in the reserve zone is based on the proposed method of Ekaka-a and Nafo [2] and Ekaka-a [3] which has been fully described above. This carefully tested numerical method is based on the hypothesis of varying a model parameter a little one-at-a-time and observing its cumulative effect on the solution trajectories or model outputs. These sensitivity values can be calculated by using the three popular mathematical norms of 1-norm, 2-norm and infinity-norm.

The current method which we have proposed in this research is more beneficial and attractive when compared with other formulae of one-at-a-time sensitivity analysis of model parameters. The difference between our present sensitivity analysis method and other methods of calculating the sensitivity of model parameters can be read in the works of [4, 5, 6, 7, 8, 9]. The results which we have obtained upon the implementation of our numerical technique are presented and discussed next.

Results and Discussions

When the model parameter β_1 whose original precise value is 0.5 is varied by 1 percent, 2 per cent, 3 percent, 4 percent and 5 percent, the new values of parameter β_1 are 0.005, 0.010, 0.015, 0.020, and 0.025. A 1 per cent variation in the model parameter β_1 will produce 147.37 cumulative percentage change in the solution trajectories using 1-norm sensitivity method, 105.94 cumulative percentage change in the solution trajectories using 2-norm sensitivity method and 87.31 cumulative percentage change in the solution trajectories using 2-norm sensitivity method using infinity-norm sensitivity method. These results are displayed at the top values of the first column of Table 1. Similarly, when the precise value of 0.4 for model parameter β_2 is varied by 1 percent, 2 percent, 3 percent, 4 percent and 5 percent, its new values are 0.004, 0.008, 0.012, 0.016, and 0.020. In this scenario, a 1 per cent variation of β_2 will produce 89.17 cumulative percentage change in the solution trajectories using a 1-norm sensitivity method, 61.24 cumulative percentage change in the solution trajectories using a 2-norm sensitivity method and 90.80 cumulative percentage change in the solution trajectories using infinity-norm sensitivity method. In the same manner, these results are displayed at the bottom values of the first column of Table 1. The sensitivity values for the 2 percent, 3 per cent, 4 percent and 5 percent variations of parameters β_1 and β_2 in terms of the 1-norm, 2-norm and infinity-norm are also displayed in Table 1. It is worth mentioning that these one-at-a-time sensitivity values in terms of the three popular mathematical norms are statistically verified as shown in Table 1.

On the basis of our chosen sensitivity method, the sensitivity values which we have calculated when the migration rate of prey species in the free zone and the migration rate of_prey species in the reserve zone are varied a little one-at-a-time as other model parameters are fixed are presented in Table 1. Without loss of generality, it is clear that the migration rate of prey species in the free zone and the migration rate of_prey species in the reserve zone whose sensitivity values are calculated in this study can be classified as relatively equally sensitive or relatively equally important.

Table 1: Calculating the sensitivity values of the migration rate of prey species in the free zone and the migration rate of prey species in the reserve zone

mn	Sensitivity values of migration rates with percentage variations					rsv	w_m	var	std
	$\beta_1 = 0.005$ $\beta_2 = 0.004$	$\beta_1 = 0.010$ $\beta_2 = 0.008$	$\beta_1 = 0.015$ $\beta_2 = 0.012$	$\beta_1 = 0.02$ $\beta_2 = 0.016$	$\beta_1 = 0.025$ $\beta_2 = 0.02$				
1-norm	147.37 89.17	143.30 87.54	139.52 85.94	136.00 84.38	132.67 82.84	14.7 6.3	137.3 84.9	33.7 6.3	5.8 2.5
2-norm	105.94 61.24	103.13 60.15	100.52 59.08	98.07 58.04	95.76 57.01	10.2 4.2	98.9 58.4	16.2 2.8	4.02 1.67
Infinity-norm	87.31 90.80	86.04 88.99	84.80 87.23	83.57 85.52	82.36 83.87	4.95 6.93	83.99 86.13	3.83 7.51	1.96 2.74

rsv represents the range of sensitivity values; w_m represents the weighted mean of sensitivity values; var represents the variance of sensitivity values; std represents the standard deviation of sensitivity values; mn represents each type of mathematical norm.

What do we learn from Table 1? Based on our choice of sensitivity hypothesis, our calculations demonstrate that both migration rates produce equally high values of the cumulative effects on the solution trajectories or model outputs when these parameters were varied a little one-at-a-time while other model parameters are fixed. These observations are consistently the same irrespective of the values of the 1-norm, 2-norm and infinity-norm ODE 45 sensitivity values. In each row of the sensitivity values for each type of mathematical norm, the top row represents the ODE 45 sensitivity values of the β_1 migration rate while the bottom row represents the ODE 45 sensitivity values of the β_2 migration rate.

Khamis et al. [1] used a deterministic model to analyse the interaction between two fish species and one predator in a patchy environment and formulated the dynamics of fishery in two homogeneous patches namely a free fishing zone and a refuge for prey reserve where fishing was prohibited. The technique of sensitivity index at a point which was proposed by Khamis et al. [1] did not look into the alternative method of tackling the parametric sensitivity of the migration rates over a time interval especially for a system of continuous nonlinear first order ordinary differential equations which evolve over time. Although, Khamis et al. [1] have also reported on the basis of their sensitivity index that the populations were more sensitive to growth, dispersal and predation rates but least sensitive to the catchability coefficient, it is also important to apply the alternative method of sensitivity analysis over a time interval to quantify the extent of the sensitivity of the migration rate parameters which Khamis et al. [1] did not analyse in their own work. Therefore, our present contribution to knowledge complements and extends the current mathematical analysis of Khamis et al. [1].

Within the mathematical sensitivity analysis literatures and as far as we know, our proposed sensitivity method over a time interval makes another useful contribution which we have not seen elsewhere. This useful result is here stated as: the 1-norm calculation of the statistical coefficient of variation (CV) which is defined as the value of the standard deviation divided by the value of the weighted mean shows that the CV of migration rate β_2 is 0.0294 while the CV of migration rate β_1 is 0.0422. Therefore, the 1-norm sensitivity for the migration rate β_2 is a better estimate than the 1-norm sensitivity for the migration rate β_1. The 2-norm calculation of CV shows that the CV of migration rate β_2 is 0.0286 while CV of migration rate β_1 is 0.0406. Therefore, the 2-norm sensitivity for the migration rate β_2 is a better estimate than the 2-norm sensitivity for the migration rate β_1. Using the same procedure, the infinity-norm calculation of CV shows that the CV of migration rate β_1 is 0.0233 while the CV of migration rate β_2 is 0.0318. In this scenario, the infinity-norm sensitivity for the migration β_1 is a better estimate than the infinity-norm sensitivity for the migration rate β_2. On the basis of these calculations, we report that the infinity-norm sensitivity measure is a better estimate for the migration rate β_1 than the 1-norm and 2-norm sensitivity measures whereas the 2-norm sensitivity measure is a better estimate for the migration rate β_2 than the 1-norm and infinity-norm sensitivity measures.

Conclusion

In this study, we have used the technique of sensitivity analysis to select the migration rate of prey species in the free zone and the migration rate of prey species in the reserve zone as two equally sensitive or equally important parameters which satisfy some standard statistical measures such as the range, weighted mean, variance and standard deviation of the sensitivity values. In respect of a further research and strengthening of knowledge-base, we recommend that these two model parameters of a prey-predator interaction with harvesting need to be estimated efficiently in order to minimise prediction uncertainty. The precise parameter values of these two parameters cannot be necessarily considered as rough estimates in terms of a further model validation analysis. Because of the significant role of the carrying capacities for this aquatic ecological model, we will be attempting to conduct the sensitivity analysis of the carrying capacities in our next study which we did not conduct in this paper.

References

[1]. S. A. Khamis, J. M. Tchuenche, M. Lukka and M. Heilio (2011), Dynamics of fisheries with prey reserve and harvesting, International Journal of Computer Mathematics, Vol. 88, No. 8, pp 1776-1802.

[2]. E.N. Ekaka-a and N.F. Nafo (2012), Parameter Ranking of Stock Market Dynamics: A Comparative Study of the Mathematical Models of Competition and Mutualistic Interactions, Scientia Africana, Vol. 11 (No. 1), pp 36-43.

[3]. E.N. Ekaka-a (2009), Computational and Mathematical Modelling of Plant Species Interactions in a Harsh Climate, PhD Thesis, University of Liverpool and University of Chester.

[4]. R.H. Gardner, D.D. Huff, R.V.O'Neill, J.B. Mankin, J. Carney and J. Jones (1980), 'Application of Error Analysis to a Marsh Hydrology Model', Water Resources Res. 16, pp 659-664.

[5]. R.V. O'Neill, R.H. Gardner and J.B. Mankin (1980), 'Analysis of Parameter Error in a Nonlinear Model', Ecol. Modelling. 8, pp 297-311.

[6]. D.J. Downing, R.H. Gardner and E.O. Hoffman (1985), 'An Examination of Response-Surface Methodologies for Uncertainty Analysis in Assessment Models', Technometrics. 27, 151-163. (See also Letter to the Editor, by R.G. Easterling and a rebuttal, Technometrics 28, pp 91-93.

[7]. D.D. Breshears (1987), Uncertainty and sensitivity analyses of simulated concentrations of radionuclides in milk. Fort Collins, CO: Colorado State University, MS Thesis, pp 1-69.

[8]. M.J. Crick, M.D. Hill and D. Charles (1987), 'The Role of Sensitivity Analysis in Assessing Uncertainty. In: Proceedings of an NEA Workshop on Uncertainty Analysis for Performance Assessments of Radioactive Waste Disposal Systems, Paris, OECD, pp 1-258.

[9]. C. Yu, J.J Cheng and A.J. Zielen (1991), 'Sensitivity Analysis of the RESRAD, a Dose Assessment Code,' Trans. Am. Nuc. Soc. 64; pp 73-74.