Journal of the Nigerian Association of Mathematical Physics
***Volume* 23 (March, 2013), pp 133 – 146**
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Three Stage Implicit Rational Runge-Kutta Schemes for Solving Discontinuous Initial Value Problems of Ordinary Differential Equations (ODES)

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Abstract

In this paper, family of Three Stage Rational Runge-Kutta schemes were derived, analysed and computerized to solve discontinous initial value problems of ordinary differential equations. Their derivation and analysis make use of Taylor and Binomial series expansion, Dalhquist stability model test equation and Pade's approximation techniques respectively. The theoretical results show that the schemes are consistent, convergent, A-stable, P-stable and A (α) stable with large interval of Absolute stability(-∞, o) and (-∞,∞).

Keywords: A-Stable,P-stable,A(α)stable,Consistent,Absolutestability,Convergent, Singular.

1.0 Introduction

The mathematical formulation of physical situations in simulation.Electrical Engineering, Control theory and Economics often lead to an initial value problems..

$$y' = f(x,y), y(x_0) = y_o \tag{1.1}$$

In which there is a pole in the solution or a singular low order derivative.
A simple example is the innocent looking initial value problem

$$y' = y^2, y(o) = 1 \quad 0 < x \le 2 \tag{1.2}$$

whose theoretical solution

$$y(x) = 1/(1-x) \tag{1.3}$$

has a simple pole at x = 1.
Another simple example is the initial value problem

$$y' = 1 + y^2, y(o) = 1 \tag{1.4}$$

in the interval $0 < x \le \pi/4$, with theoretical solution

$$y(x) = \tan(x + \pi/4) \tag{1.5}$$

such problems are classified as discontinuous initial value problems.
Researchers .that have contributed to the development of these areas of ODEs include Cash [1], Enright [2], Burrage Butcher and Chipman[3],Gafney[4], Norsett[5], Alexander[6], Kap and Kentrop[7] Roger[8] , Lubitch et al[9]. The consequences of these contributions lead to an introduction of the Rational one-step method of the form

$$y_{n+1} = \frac{y_n^2}{y_n - hf_n} \tag{1.6}$$

called Inverse Euler's scheme [10]. This perhaps made Hong-Yuafu [11] to propose a more general Rational Runge-Kutta method of the form

$$y_{n+1} = \frac{y_n + \sum_{i=1}^{R} W_i K_i}{1 + y_n \sum_{i=1}^{R} V_i H_i} \tag{1.7}$$

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where ,

$$K_1 = hf(x_n, y_n)$$

$$K_i = hf(x_n + c_i h, y_n + \sum_{j=1}^{R} a_{ij} K_j)$$

$$H_1 = hg(x_n, z_n)$$

$$H_i = hg\left(x_n + d_i h,\ z_n + \sum_{j=1}^{R} b_{ij} H_j \right)$$

with $$g(x_n, z_n) = -z_n^2 f(x_n, y_n)$$

and $$z_n = 1/y_n \qquad (1.8)$$

In his development, he considered the explicit family of the method that is the case $a_{ij} = b_{ij} = 0$ for $j \geq i$ and developed family of order one, two and three of the methods. Okunbor [12] also develop the higher stage of the method. This prompted Babatola [13] to consider the implicit family of the stages one and two. In this work, we shall consider the formular of stage three Implicit Rational Runge–Kutta. Next section discusses the development of the new schemes.

2. 0 The Development of the Proposed Schemes

Recall from equation (1.7) that an R-stage Implicit Rational Runge-Kutta scheme is

$$y_{n+1} = \frac{y_n + \sum_{i=1}^{R} W_i K_i}{1 + y_n \sum_{i=1}^{R} V_i H_i} \qquad (2.1)$$

where,

$$K_i = hf\left(x_n + c_i h,\ y_n + \sum_{i=1}^{R} a_{ij} K_j \right)$$

$$H_i = hg\left(x_n + d_i h,\ z_n + \sum_{j=1}^{R} b_{ij} H_j \right) \qquad (2.2)$$

with

$$g(x_n, z_n) = -z_n^2 f(x_n, y_n) \qquad (2.3)$$

and

$$z_n = 1/y_n$$

with the constraints

$$c_i = \sum_{j=1}^{i} a_{ij}, i = 1(1)R \qquad (2.4)$$

$$d_i = \sum_{j=1}^{i} b_{ij}, \quad i = 1(1)R \qquad (2.5)$$

and h is the step size or grid spacing. These constraints ensure the consistency of the schemes. The parameter V_i, W_i, C_i, d_i, a_{ij} and b_{ij} are to be determine from the system on non-linear equation generated by adopting the following steps;

(i) Adopting the Taylor and Binomial series expansion and Insert the series of expansion into equation (2.1)

(ii) Compare the final expansion with the Taylor series expansion of y_{n+1} about (x_n, y_n) in the powers y_{n+1} of h.

The numbers of parameters normally exceeds the number of equations, but in the spirit of King[14], Gill[15] and Blum[16] these parameters are chosen to ensure that (one or more of the following conditions are satisfied).

1. Adequate order of accuracy of the scheme is achieved
2. Minimum bound of local truncation error exists.
3. The method has maximize interval of absolute stability .
4. Minimize computer storage facilities are utilized.

2.1.1 Three – Stage Schemes

By equation (2.1), the general three-stage Implicit Rational Runge-Kutta scheme is of the form

$$y_{n+1} = \frac{y_n + \sum_{i=1}^{3} W_i K_i}{1 + y_n \sum_{i=1}^{3} V_i H_i} \qquad (2.6)$$

where,

$$\mathrm{K_i} = hf\left(x_n + \mathrm{c}_1 h,\ \mathrm{y_n} + \sum_{j=1}^{3} a_{ij} K_j \right),\ \mathrm{i=1(1)3} \qquad (2.7)$$

$$\mathrm{H_i} = hg\left(x_n + d_1 h,\ \mathrm{z_n} + \sum_{j=1}^{3} b_{1j} H_j \right), i = 1(1)3 \qquad (2.8)$$

with $g(\mathrm{x_n}\ \mathrm{z_n}) = -\mathrm{z}_n^2 \mathrm{f}(\mathrm{x_n}, y_n)$ (2.9)

and $\mathrm{z_n} = 1/y_n$ (2.10)

with the constraints

$$\mathrm{c_i} = \sum_{j=1}^{3} a_{ij} \qquad (2.11)$$

$$\mathrm{d_i} = \sum_{j=1}^{3} b_{ij} \qquad (2.12)$$

Adopting Binomial expansion theorem on the right hand side of equation (2.6) and ignoring higher order terms, we obtained

$$\mathrm{y_{n+1}} = y_n + \sum_{i=1}^{3} \mathrm{W_i} K_i - \mathrm{y_n^2} \sum_{\mathrm{i=1}}^{3} \mathrm{V_i} H_i + (\text{higher orderr term}) \qquad (2.13)$$

Also the Taylor series expansion of y_{n+1} about y_n gives

$$\mathrm{y_{n+1}} = y_n + \mathrm{hy'_n} + \frac{\mathrm{h}^2}{2!} y''_n + \frac{\mathrm{h}^3}{3!} \mathrm{y'''_n} + \frac{h^4}{4!} \mathrm{y}_n^{(\mathrm{IV})} + \frac{\mathrm{h}^5}{5!} \mathrm{y_n^{(v)}} + \frac{\mathrm{h}^6}{6!} \mathrm{y_n^{(vi)}} + 0\mathrm{h}^7 \qquad (2.14)$$

Now

$$\mathrm{y'_n} = \mathrm{f(x_n, y_n)} = \mathrm{f_n}$$
$$y''_n = \mathrm{f_x} + f_n \mathrm{f_y} = \mathrm{Df_n}$$
$$\mathrm{y'''_n} = \mathrm{D^2 f_n} + f_y D f_n$$
$$\mathrm{y_n^{(iv)}} = \mathrm{D^3 f_n} + f_y D^2 f_n + 3Df_n Df_y + fy^2 Df_n$$
$$y_n^{(v)} = D^4 f_n + f_y D^3 f_n + f_{yy} D^2 f_n + 6Df_y D^2 f_n + 3Df_n Df_y + f_y{}^2 D^3 f_y \qquad (2.15)$$

where

$$\mathrm{D}^4 f_n = f_{xxxx} + 4 f_n f_{xxxy} + 6 f_n^2 f_{xxyy} + 4 f_n^3 f_{xyyyy} + f_n^4 f_{yyyy}$$
$$\mathrm{D}^3 f_n = f_{xxx} + 3 f_n f_{xxy} + 3 f_n^2 f_{xxy} + 3 f_n^2 f_{xyy} + f_n^3 f_{yyy}$$
$$Df_y = f_{xy} + f_n f_{yy} + f_n f_{yy} + f_y^2 \qquad (2.16)$$
$$D^2 f_n = f_{xx} + 2 f_n f_{xy} + f_n{}^2 f_{yy}$$

Substitute equation (2.15) into equation (2.14) we have

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2!}Df_n + \frac{h^3}{6!}\left(D^2 f_n + f_y Df_n\right) + \frac{h^4}{24!}\left(D^3 f_n + f_y D^2 f_n + 3Df_n Df_y + f_y^2 Df_n\right) + \frac{h^5}{120}$$

$$\left(D^4 f_n + f_y D^3 f_n + f_{yy} D^2 f_n + 6Df_y D^2 f_n +\right) Df_n Df_y + f_y{}^2 D^3 f_y) \qquad (2.17)$$

Similarly expand K_i , $i = 1, 2, 3$ about (x_n, y_n) we have,

$$K_i = h\left(f_n + \left(c_i hf_x + (a_{11}k_1 + a_{12}k_2 + a_{13}k_3) f_y\right)\right) + \frac{1}{2}\left(c_i^2 h^2 f_{xx} + 2c_1 h((a_{11}k_1 + a_{12}k_2 + a_{13}k_3\right) f_{xy} +$$

$$(a_{11}k_1 + a_{12}k_2 + a_{13}k_3)^2 f_{yy} + 1/6\left(c_i^3 h^3 f_{xxx} + 3c_i^2 h^2 (a_1 k_1 + a_{i2}k_2 + a_{i3}k_3) f_{xxy} + 3c_i h_2\right.$$

$$(a_{11}k_1 + a_{12}k_2 + a_{13}k_3)^2 \ f_{xyy} +$$

$$(a_{11}k_1 + a_{12}k_2 + a_{13}k_3)^3 f_{yyy}\Big] + 1/24\left(c_i^4 h^4 f_{xxxx} + 4\left(c_i^3 h^3 (a_{i1}k_i + a_{12}k_2 + a_{12}k_3\right) f_{xxxy} + 6c_i^2 h^2\right.$$

$$(a_{11}k_1 + a_{12}k_2 + a_{13}k_3)^2 \ f_{xxyy} + 4c_i h\ (a_{11}k_1 + a_{12}k_2 + a_{13}k_3)^3 f_{xyyy} + \qquad (a_{11}k_1 + a_{12}k_2 + a_{13}k_3)^4 f_{y\,yyy}$$

(2.18)

By collecting coefficients of terms in the equal powers of h becomes

$$K_i = hA_i + h^2 B_i + h^3 E_i + h^4 D_1 + h^5 P_i, \mathrm{i} = 1,2,3 \qquad (2.19)$$

where

$A_i = f_n$

$$E_i = (a_{i1}B_1 + a_{i2}B_2 + a_{i3}B_3) f_y \ + 1/2 c_i^2 f_{xx} + c_i (a_{i1}A_1 + a_{i2}A_2 + a_{i3}A_3) f_{xy} +$$

$$(a_{i1}A_1 + a_{i2}A_2 + a_{i3}A_3) f_{yy}$$

$$= (a_{i1}c_1 + a_{i2}c_{2n} + a_{i3}c_3) Df_n + 1/2 c_i^2 D^2 f_n$$

$$B_i = c_i f_x + (a_{i1}A_1 + a_{i2}A_2 + a_{i3}A_3) f_y = c_i Df_n$$

$$D_i = \left[a_{i1}(a_{i1}c_1 + a_{i2}c_2 + a_{i3}c_3) + a_{i2}(a_{21}c_1 + a_{22}c_2 + a_{23}c_3) + a_{i3}(a_{31}c_1 + a_{32}c_2 + a_{33}c_3)\right] f_y{}^2 Df_n +$$

$$1/2\left(a_{i1}c_1^2 + a_{i2}c_2^2 + a_{i3}c_3^2\right) f_y D^2 f_n + 1/2\left(a_{i1}c_1^2 + a_{i2}c_2^2 + a_{i3}c_{3\,3}^2\right) f_y D^2 f_n + 1/6\, c_1^3 D^3 f_n$$

$$P_i = a_{i1}(a_{11}c_1 + a_{12}c_2 + a_{13}c_3)(a_{11} + a_{21} + a_{31}) + (a_{12} + a_{22} + a_{32})(a_{21}c_1 + a_{22}c_2 + a_{23}c_3) f_{y^2} Df_n$$

$$f_{yy} + 1/2 (c_1 + c_2 + c_3)(a_{11}c_1 + a_{12}c_2 + a_{13}c_3) Df_n Df_y + 1/2\left(a_{i1}c_1^2 + a_{i2}c_2^2 + a_{i3}c_3^2\right) + 1/6\left(c_1^3 + c_2^3 + c_3^3\right) D^3 f_n$$

In a

$$+ \left[(c_1 + c_2 + c_3)(a_{11}c_1 + a_{21}c_1 + a_{31})\right] f_y Df_n + 1/2\left(c_1^2 + c_2^2 + c_3^2\right) D^2 f_n + (a_{11}c_1 + a_{21}c_2 + a_{31}c_3) Df_n$$

$$+ (a_{11} + a_{22} + a_{33}) f_n + 1/6\left(c_1^3 + c_2^3 + c_3^3\right) D^3 f_n + 1/24\, c_1^4 D^4 f_n \qquad (2.20)$$

similar manner expanding H_i, i = 1, 2 , 3 about (x_n, z_n)

$H_i = hN_i + h^2M_i + h^2R_i + h^4L_i + h^5G_i + 0h^6$ (2.21)

where

$N_I = g_n$

$M_I = d_i gx + (b_i, N_i + b_{i2} N_2 + b_3N_3) = d_i Dg_n$

$$R_i = (b_{i1}M_1 + b_{i2}M_2 + b_{i3}M_3) + 1/2 d_i^2 g_{xx} + d_i (b_{i1}N_1 + b_{i2}N_2 + b_{i3}N_3) g_{xz}$$

$$+ (b_{ii1}N_1 + b_{i2}N_2 + b_{i3}N_3) g_{xz} + (b_{ii1}d_1 + b_{i2}d_2 + b_{i3}d_3) g_{zz} Dg_n + 1/2 d_i D^2 g_n$$

$$L_i = (b_{i1}R_1 + b_{i2}R_2 + b_{i3}R_3)g_Z +_i (b_{i1}M_1 + b_{i2}M_2 + b_{i3}R_3)g_{xz}$$
$$+ (b_{ii1}N_1 + b_{i2}N_2 + b_{i3}N_3)(b_{ii1}M_1 + b_{i2}M_2 + b_{i3}M_3)g_{zz} + \tfrac{1}{6} d_i g_{xxx}$$
$$+ \tfrac{1}{2} d_1^2 (b_{ii1}N_1 + b_{i2}N_2 + b_{i3}N_3)g_{xxz} + \tfrac{1}{2} d_i (b_{i1}N_1 + b_{i2}N_2 + b_{i3}N_3)g_{xxz} +$$
$$\tfrac{1}{6}(b_{ii1}N_1 + b_{i2}N_2 + b_{i3}N_3)g_{zzz}$$
$$= b_{i1}(b_{i1}d_1 + b_{i2}d_2 + b_{i3}d_3) + b_{i2}(b_{21}d_1 + b_{22}d_2 + b_{23}d_3)g_z^2 Dg_n + d_1(b_{i1}d_1 + b_{i2}d_2 + b_{i3}d_3)Dg_n$$
$$+ \tfrac{1}{2}(b_{i1}d_{_11}^2 + b_{i2}d_2^2 + b_{i3}d_3^2)D^2 g_n + \tfrac{1}{6} d_1^3 D^3 g_n$$
$$G_i = [b_{11}(b_{i1}d_1 + b_{i2}d_2 + b_{i3}d_3)(b_{11} + b_{21} + b_{22} + b_{31}) + (b_{12} + b_{22} + b_{32})(b_{21}d_1 + b_{22}d_2 + b_{23}d_3)]g_y Dg_n g_{zz}$$
$$+ \tfrac{1}{2}(d_1^2(b_{11}d_1 + b_{12}d_2 + b_{13}d_3))Dg_n, Dg_z + (b_{11}d_1^2 + b_{12}d_2^2 + b_3 d_3^2) + \tfrac{1}{6}(d_1^2 + d_2^2 + d_3^2)D^2 g_n$$
$$+ (d_1^2 + d_2^2 + d_3^2)(b_{11}d_1 + b_{12}d_2 + b_{13}d_3)g_z Dg_n + \tfrac{1}{2}(d_1^2 + d_2^2 + d_3^2)D^2 g_n + (b_{11}d_1 + b_{22}d_2 + b_{23}d_3)D^3 g_n$$
$$+ \tfrac{1}{6}(d_1^3 + d_2^3 + d_3^3) + \tfrac{1}{24} d_1^4 D^4 g_n \qquad (2.22)$$

Expressing g and its partial derivatives in terms of f and its partial derivatives to facilitates the comparison of coefficients. That is,

$$g_n = \frac{-f_n}{y_n^2},\ \mathrm{g_x} = \frac{-f_x}{y_n^2},\ \mathrm{g_{xx}} = \frac{-f_{xx}}{y_n^2}$$
$$g_{xxx} = \frac{-f_{xxxx}}{y_n^2}, \qquad g_{xxxx} = \frac{-f_{xxxx}}{y_n^2}$$
$$\mathrm{g_z} = \frac{-2f_n}{y_n} + f_y,\ \ \mathrm{g_{xz}} = \frac{-2f_x}{y_n} + f_{xy}$$
$$\mathrm{g_{xxz}} = \frac{-2f_{xx}}{y_n} + f_{xxy},\ \ \mathrm{g_{xxxz}} = \frac{-2f_{xx}}{y_2^n} + f_{xxxy},$$
$$\mathrm{g_{xzzz}} = -4y_n^2 f_{xyy} + 6y_n^2 f_{xyy} + y_n^2 f_{xyy}$$
$$g_{zzz} = 4y_n^2 f_y + 6y_n^2 + y_n^4 f_{yyy}$$
$$g_{xzz} = -2f_x + y_n^2 + f_{xyy},$$
$$\mathrm{g_{xzzz}} = -4y_n^2 f_{yy} + 6y_n^2 f_{yyy} + y_n^4 f_{yyyy} \qquad (2.23)$$

Substitute (2.23) into (2.22) obtained

$$\mathrm{Ni} = \frac{-f_n}{y_n^2},\ \mathrm{M_i} = \frac{\mathrm{-d_i}}{\mathrm{y_n^2}}\left(Df_n + \frac{\mathrm{2f_n^2}}{\mathrm{y_{nn}^2}}\right)$$
$$\mathrm{Ri} = \frac{-di^2}{y_n^2}\left[\left(\frac{-2f_n}{y_n} + f_y\right)\left(\mathrm{Df_n} + \frac{f_n^2}{y_n}\right)\right] + \tfrac{1}{2}\left(D^2 f_n - \frac{Df_n}{y_n}\left(\frac{f_n^2}{y_n} + f_x\right)\right]$$
$$L_i = (b_{i1}R_1 + b_{i2}R_2 + b_{i3}R_3)\left(\frac{-2f_n}{y_n} + f_y\right) +$$
$$d_i(b_{ii1}M_1 + b_{i2}M_2 + b_{i3}M_3)\left(\frac{-2f_x}{y_n} + f_{xy}\right) + \tfrac{1}{6}\frac{d_1^3}{y_n^2}\left(D^3 f_n + \frac{6f_n f_{xx} - 6f_n f_x}{y_n} - 4f_n^3 f_y - f_n^3 f_{yy}\right)$$

$$G_i = (b_{i1}L_1 + b_{i2}L_2 + b_{i3}L_3)\left(\frac{-2f_n}{y_n^2} + f_y\right) + d_i(b_{21}R_1 + b_{22}R_2 + b_{21}R_3)\left(\frac{-2f_n}{y_n^2} + f_{xy}\right)$$

$$+ \tfrac{1}{24}\frac{d_1^4}{y_n^2}\left(D^4 f_n + f_n f_{xxx} - 12 f_n^2 f_{xx} + 16 f_n^2 f_{xxx} + 16 f_n^3 f_{xyy} + 24 f_n^2 f_{xyy} + 4 f_n^4 f_{yy} + 6 f_n^4 f_{yyyy}\right) \quad (2.24a)$$

Recalling (2.13)

$$y_{n+1} = y_n + (W_1K_1 + W_2K_2 + W_3K_3) - y_n^2(V_1H_1 + V_2H_2 + V_3H_3) + (higher\ order)(2.24b)$$

Now using (2.19) and (2.21), we obtain

$$y_{n+1} = y_n + h\left[(W_1A_1 + W_2A_2 + W_3A_3)y_n^2(V_1N_1 + V_2N_2 + V_3N_3)\right] + h^2(W_1B_1 + W_2B_2 + W_3B_3) - y_n^2$$
$$(V_1M_1 + V_2M_2 + V_3M_3) + h^2\left[(W_1B_1 + W_2B_2 + W_3B_3) - y_n^2(V_1M_1 + V_2M_2 + V_3M_3) + \right.$$
$$h^3(W_1E_1 + W_2E_2 + W_3E_3) - y_n^2(V_1N_1 + V_2N_2 + V_3N_3) + h^4(W_1D_1 + W_2D_2 + W_3D_3) - y_n^2$$
$$(V_1L_1 + V_2L_2 + V_3L_3) + h^5(W_1P_1 + W_2P_2 + W_3P_3)y_n^2(V_1G_1 + V_2G_2 + V_3G_3) + 0h^6$$

(2.25)

Comparing the coefficient of h, h^2, h^3, h^4, h^5 in equation (2.25) and (2.17) we obtain the following systems of equation for family of third stage schemes of order five

$$W_1+W_2+W_3+V_1+V_2+V_3 = 1$$
$$W_1C_1 + W_2C_2 + W_3C_3V_1d_1 + V_2d_2 + V_3d_3 = \tfrac{1}{2}$$
$$W_1c_1^2 + W_2c_2^3 + W_3c_3^3 + V_1d_1^2 + V_2d_2^2 + V_3d_3^2 = \tfrac{1}{3}$$
$$W_1c_1^2 + W_2c_2^3 + W_3c_3^3 + V_1d_1^3 + V_2d_2^3 + V_3d_3^3 = \tfrac{1}{4}$$
$$W_1(a_{11}c_1 + a_{12}c_2 + a_{13}c_3) + W_2(a_{21}c_1 + a_{22}c_2 + a_{23}c_3)$$
$$+ W_3(a_{31}c_1 + a_{32}c_2 + a_{23}c_3) + V_1(b_{11}d_1 + b_{12}d_2 + b_{13}d_3)$$
$$+ V_2(b_{21}d_1 + b_{22}d_2 + b_{23}d_3) + V_3(b_{31}d_1 + b_{32}d_2 + b_{33}d_3) = \tfrac{1}{6}$$
$$W_1C_1(a_{11}c_1 + a_{12}c_2 + a_{13}c_3) + W_2C_2(a_{21}c_1 + a_{22}c_2 + a_{23}c_3)$$
$$+ W_3C_3(a_{31}c_1 + a_{22}c_1^2 + a_{32}c_2^2 + a_{33}c_3^2) +$$
$$V_1d_1(b_{11}d_1 + b_{12}d_2 + b_{13}d_3) + V_2d_2(b_{11}d_1 + b_{22}d_2 + b_{23}d_3)$$
$$+ V_3d_3(b_{31}d_1 + b_{32}d_2 + b_{33}d_3) = \tfrac{1}{2}$$
$$(W_1a_{11} + W_2a_{21} + W_3b_{31})(a_{11}c_1 + a_{12}c_2 + a_{13}c_3) +$$
$$(W_1a_{12} + W_2a_{22} + W_3a_{23})(a_{21}c_1 + a_{22}c_2 + a_{23}c_3) +$$
$$W_3c_3(a_{31}c_1 + a_{32}c_2 + a_{33}c_3) + (V_1b_{11} + V_2b_{21} + V_3b_{31})(b_{11}d_1 + b_{13}d_3 + b_{13}d_3) +$$
$$(V_1b_{12} + V_2b_{22} + V_3b_{32})(b_{21}d_1 + b_{22}d_2 + b_{23}d_3) +$$
$$(V_1b_{13} + V_2b_{23} + V_3b_{33})(b_{31}d_1 + b_{32}d_2 + b_{33}d_3) = \tfrac{1}{24} \quad (2.26))$$

Subject to constraints

$$b_{11} + b_{12} + b_{13} = d_1$$
$$b_{21} + b_{22} + b_{23} = d_2$$
$$b_{31} + b_{32} + b_{33} = d_3$$

and

$$a_{11} + a_{12} + a_{13} = c_1$$
$$a_{21} + a_{22} + a_{23} = c_2$$
$$a_{31} + a_{32} + a_{33} = c_3 \quad (2.27))$$

Solving these equations for the parameters W_1, W_2, W_3, V_1, V_2, V_3, a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}, b_1, b_{12}, b_{13}, b_{21}, b_{22}, b_{23}, b_{31}, b_{32}, b_{33}, d_1, d_2, d_3, c_1, c_2, c_3, we get;

(1) $V_1 = W_1 = \tfrac{1}{18}$. $V_2 = W_2 = \dfrac{16 + \sqrt{6}}{72}$

$$W_3 = V_3 = \frac{16-\sqrt{6}}{72},\ \mathrm{C}_1 = d_1 = 0$$

$$C_2 = d_2 = \frac{6-\sqrt{6}}{20}, \mathrm{C}_3 = d_3 = 6+\frac{\sqrt{6}}{20}$$

$$a_{21} = b_{11} = a_{12} = b_{12} = a_{13} = b_{13} = 0$$

$$b_{21} = a_{21} = \frac{9+\sqrt{6}}{150}; \mathrm{b}_{22} = a_{22} = \frac{24+\sqrt{6}}{240}$$

$$b_{23} = a_{23} = \frac{168-73\sqrt{6}}{1200}; \mathrm{a}_{31} = b_3 = \frac{9+\sqrt{6}}{150}$$

$$b_{32} = a_{32} = \frac{168+73\sqrt{6}}{1200}; \mathrm{b}_{33} = a_{33} = \frac{24-\sqrt{6}}{240}$$

Yielding a third stage method of order five of the form

$$\mathrm{y}_{n+1} = \frac{y_n + \frac{1}{72}\left(4K_1 + \left(16+\sqrt{6}\right)K_2 + 2\left(16-\sqrt{6}\right)K_3\right.}{1+\frac{y_n}{72}\left(4H_1 + \left(16+\sqrt{6}\right)H_2 + \left(16-\sqrt{6}\right)H_3\right.} \tag{2.28}$$

where

$$K_1 = hf\,(x_n, y_n)$$

$$K_2 = hf\left(x_n + \frac{\left(6-\sqrt{6}\right)h}{20}, y_n + \frac{\left(9+\sqrt{6}\right)}{150}K_1 + \left(\frac{24+\sqrt{6}}{240}\right)K_2 + \left(\frac{168-73\sqrt{6}}{1200}\right)K_3\right)$$

$$K_3 = hf\left(x_n + \left(\frac{6+\sqrt{6}}{20}\right)h,\ \mathrm{y}_n + \left(\frac{9+\sqrt{6}}{150}\right)K_1 + \left(\frac{9-\sqrt{6}}{150}\right)K_2\left(\frac{24-\sqrt{6}}{240}\right)K_3\right)$$

$$H_1 = hg(x_n, \mathrm{z}_n)$$

$$H_2 = hg\left(x_n + \frac{\left(6-\sqrt{6}\right)h}{20}, z_n + \frac{\left(9+\sqrt{6}\right)}{150}H_1 + \left(\frac{24-\sqrt{6}}{240}\right)H_2 + \left(\frac{24-\sqrt{6}}{240}\right)H_3\right)$$

$$H_3 = hg\left(x_n + \left(\frac{6-\sqrt{6}}{20}\right)h,\ \mathrm{z}_n + \left(\frac{9-\sqrt{6}}{150}\right)H_1 + \left(\frac{168+73\sqrt{6}}{1200}\right)H_2 + \left(\frac{24-\sqrt{6}}{120}\right)H_3\right) \tag{2.29}$$

For the case

(2) $\mathrm{V}_1 = W_1 = \frac{16-\sqrt{6}}{72},\ \mathrm{W}_2 = V_2 = \frac{16+\sqrt{6}}{72}-, \mathrm{V}_3 = W_3 = 1/18$

$$\mathrm{c}_1 = d_1 = \frac{4-\sqrt{6}}{20},\ \mathrm{c}_2 = d_2 = \frac{4+\sqrt{6}}{20}, c_3 = d_3 = 1/2$$

$$\mathrm{b}_{11} = a_{11} = \frac{24-\sqrt{6}}{120},\ \mathrm{b}_{12} = a_{12} = \frac{24-11\sqrt{6}}{240}$$

$$\mathrm{b}_{13} = a_{13} = b_{23} = a_{23} = b_{33} = a_{33} = 0$$

$$\mathrm{b}_{21} = a_{21} = \frac{24+\sqrt{6}}{240},\ \mathrm{b}_{31} = a_{12} = \frac{6-\sqrt{6}}{24} = a_{31}$$

$$a_{32} = b_{32} = \frac{6+\sqrt{6}}{24}$$

We get another 3 – stage method of order five as:

$$y_{n+1} = y_n + \frac{\frac{1}{72}\left(\left(16-\sqrt{6}\right)K_1 + \left(16+\sqrt{6}\right)K_2 + 4K_3\right)}{1+\frac{y_n}{72}\left(\left(16-\sqrt{6}\right)H_1 + \left(16+\sqrt{6}\right)H_2 + 4H_3\right)} \qquad (2.30)$$

where

$$K_1 = hf\left(x_n + \left(\frac{4-\sqrt{6}}{20}\right)h,\ y_n + \left(\frac{24-\sqrt{6}}{240}\right)K_1 + \left(\frac{24-11\sqrt{6}}{24}\right)K_2\right)$$

$$K_2 = hf\left(x_n + \left(\frac{4-\sqrt{6}}{20}\right)h,\ y_n + \left(\frac{24-\sqrt{6}}{240}\right)K_1 - \left(\frac{24-\sqrt{6}}{120}\right)K_2\right)$$

$$K_3 = hf\left(x_n + \frac{h}{2}, y_n + \left(\frac{6-\sqrt{6}}{24}\right)K_1 + \left(\frac{6+\sqrt{6}}{24}\right)K_2\right)$$

$$H_1 = hg\left(x_n + \frac{\left(4-\sqrt{6}\right)h}{20}, z_n + \frac{\left(24-\sqrt{6}\right)}{240}H_1 + \left(\frac{24-1K\sqrt{6}}{24}\right)H_2\right)$$

$$H_2 = hg\left(x_n + \frac{\left(4+\sqrt{6}\right)h}{20}, y_n + \frac{\left(24-\sqrt{6}\right)}{240}H_1 + \left(\frac{24-\sqrt{6}}{120}\right)H_2\right) \qquad (2.31)$$

$$H_3 = hg\left(x_n + \frac{h}{2},\ Z_n + \frac{\left(24-\sqrt{6}\right),}{240}, H_1 + \frac{\left(24-11\sqrt{6}\right)}{24}H_2\right)$$

Set of non-linear system of equations for solving third stage scheme of order six are obtained as

$$W_1 + W_2 + W_3 + V_1 + V_2 + V_3 = 1$$

$$W_1c_1 + W_2c_2 + W_3c_3 + V_1d_1 + V_2d_2 + V_3d_3 = \tfrac{1}{2}$$

$$W_1c_1^2 + W_2c_2^2 + W_3c_3^2 + V_1d_1^2 + V_2d_2^2 + V_3d_2^3 = \tfrac{1}{3}$$

$$W_1\ c_1^3 + W_2\ c_2^3 + W_3c_3^3 + V_1d_1^3 + V_2d_2^3 + V_3d_3^3 = \tfrac{1}{4}$$

$$W_1\left(a_{11}c_1^2 + a_{12}c_2^2 + a_{13}c_3^2\right) + W_2\left(a_{21}c_1^2 + a_{22}c_2^2 + a_{23}c_3^2\right)$$
$$W_3\left(a_{31}c_1^2 + a_{32}c_2^2 + a_{33}c_3^2\right) + V_1\left(b_{11}d_1^2 + b_{12}d_2^2 + b_{13}d_3^2\right)$$
$$V_2\left(b_{21}d_1^2 + b_{22}d_2^2 + b_{23}d_3^2\right) + V_3\left(b_{31}d_1^2 + b_{32}d_2^2 + b_{33}d_3^2\right) = \tfrac{1}{12}$$

$$W_1c_1\left(a_{11}c_1 + a_{12}c_2 + a_{13}c_3\right) + W_2c_2\left(a_{21}c_1 + a_{22}c_2 + a_{23}c_3\right)$$
$$W_3c_3\left(a_{31}c_1 + a_{32}c_2 + a_{33}c_3\right) + V_1d_1\left(b_{11}d_1 + b_{12}d_2 + b_{13}d_3\right)$$
$$+V_2d_2\left(b_{21}d_1 + b_{22}d_1 + b_{23}d_3\right) + V_3d_3\left(b_{31}d_1 + b_{32}d_2 + b_{33}d_3\right) = \tfrac{1}{6}$$

$$W_1\left(a_{11}d_1^2 + a_{12}d_2^2 + a_{13}d_3^2\right) + W_2\left(a_{21}c_1^2 + a_{22}c_2^2 + a_{23}c_3^2\right)$$
$$W_3\left(a_{31}c_1^2 + a_{32}c_2^2 + a_{33}c_3^2\right) + V_1\left(b_{11}d_1^2 + b_{12}d_2^2 + b_{13}d_3^2\right) \qquad (2.\ 32)$$
$$V_2\left(b_{21}d_1^2 + b_{22}d_2^2 + b_{23}d_3^2\right) + V_3\left(b_{31}d_1^2 + b_{32}d_2^2 + b_{33}d_3^2\right) = \tfrac{1}{48}$$

With the constraints in equation (2.27) solving equations (2.32) and (2.27) we obtained a six order method of the form

$$y_{n+1} = \frac{y_n + \frac{1}{36}(5K_1 + 8K_2 + 5K_3)}{1 + \frac{y_n}{36}(5K_1 + 8H_2 + 5H_3)} \qquad (2.33)$$

where

$$K_1 = hf\left(x_n + \left(\frac{5-\sqrt{15}}{20}\right)h, \mathrm{y_n} + \frac{5}{72}K_1 + \left(\frac{10-3\sqrt{15}}{90}\right)K_2 + \frac{25-6\sqrt{15}}{360}K_3\right)$$

$$K_2 = hf\left(x_n + \frac{h}{4}, \mathrm{y_n} + \left(\frac{10`-3\sqrt{15}}{144}\right)K_1 + \frac{2}{18}K_2 + \left(\frac{10-3\sqrt{15}}{144}\right)K_3\right)$$

$$K_3 = hf\left(x_n + \left(\frac{15-\sqrt{15}}{20}\right)h, \mathrm{y_n} + \left(\frac{25-3\sqrt{15}}{360}\right)K_1 + \frac{10=3\sqrt{5}}{90}K_2 + \frac{5}{18})K_3\right)$$

$$H_1 = hg\left(x_n + \frac{(5-\sqrt{15})h}{20}, y_n + \frac{5}{72}H_1 + \frac{(10-3\sqrt{15})}{90}H_2 + \left(\frac{25-6\sqrt{15}}{360}\right)H_3\right)$$

$$H_2 = hg\left(x_n + \frac{h}{4}, \mathrm{Z_n} + \frac{(10-3\sqrt{15})}{144}, H_1 + \frac{2}{18}H_2 + \frac{(10-3\sqrt{15})}{144}\right)H_3\right) \qquad (2.34)$$

$$H_3 = hg\left(x_n + \frac{(15-\sqrt{15})}{20}h\ , z_n + \frac{(25-6\sqrt{15})}{360}H_1 + \frac{(10-3\sqrt{15})}{90}H_2 + \frac{5}{18}H_3\right)\right)$$

With the case

$$\mathrm{V_1} = W_1 = \frac{5}{36}, \mathrm{V_2} = W_2 = \frac{4}{18}, \mathrm{V_3} = W_3 = \frac{5}{36}$$

$$d_1 = C_1 = \frac{10-3\sqrt{15}}{90}, C_2 = d_2 = \frac{1}{4}., \ \mathrm{C_3} = d_3 = \frac{5-\sqrt{5}}{20}$$

$$b_{11} = a_{11} = 5/72, \quad \mathrm{b_{12}} = a_{13} = \frac{10-3\sqrt{15}}{45},$$

$$a_{13} = b_{13} = \frac{25-6\sqrt{15}}{360}, \ \mathrm{a_{21}} = b_2 = \frac{10+3\sqrt{5}}{144},$$

$$b_{22} = a_{22} = \frac{1}{9}, \ \mathrm{b_{23}} = a_{23} = \frac{10-3\sqrt{5}}{144},$$

$$a_{31} = b_{31} = \frac{25-6\sqrt{15}}{360}, \ \mathrm{a_{32}} = b_{32} = \frac{10-3\sqrt{15.}}{90}, \mathrm{b_{33}} = a_{38} = \frac{5}{72}$$

3.0 Analysis of the Basic Properties

The basic properties required of a good computational method include error, convergence and stability. Therefore in this section, we shall consider the analysis of error, convergence and stability properties of the proposed schemes

3.1 Analysis of Error.

Errors of numerical approximation majorly classified into Round off errors, Truncation and discretization.
Round off errors is an error introduced as a result of the computing devices.
Mathematically, it can be expressed as

$$r_{n+1} = y_{n+1} - \mathrm{P}_{n+1} \quad (3.1)$$

Where y_{n+1} is the expected solution of the difference equation (2.6) while P_{n+1} is the computer output at the $(n+1)^{th}$ iteration. Truncation error is the error introduced as a result of ignoring some of the higher terms of the power series (Taylor and Binomial series expansion) during the development of the new formulas.

$$T_{n+1} = y(x_{n+1}) - \frac{y(x_n) + \sum_{i=1}^{3} W_i K_i}{1 + y(x_n)\sum_{i=1}^{3} V_i H_i} \quad (3.2)$$

$$K_i = hf(x_n + c_i h, y_n + \sum_{j=1}^{3} a_{ij} K_j), \text{i=1,2,3}$$

$$H_i = hg(x_n + d_i h, z_n + \sum_{j=1}^{3} b_{ij} H_j), \text{i=1,2,3} \quad (3.3)$$

Discretization error is an error introduced as a result of transforming a differential equation (1.1) into difference equation (2.6).
Mathematically,

$$e_{n+1} = y_{n+1} - y(x_{n+1}) \quad (3.4)$$

3.2 <u>The Convergence Property</u>

The numerical scheme (2.6) for solving ODEs (1.1) is said to be convergent, if the numerical approximation y_{n+1}, that is generated by it tends to the exact solution $y(x_{n+1})$ of the ODEs (1.1) as the step size tends to zero.
That is

$$\lim_{n->\infty} (y(\mathrm{x}_{n+1}) - y_{n+1}) = 0 \quad (3.5)$$

To analyze he convergent of the proposed scheme, we consider the following theorem which we state without proof.
Theorem: Let $\{e_j\}$, $\mathrm{j} = 0(1)\mathrm{n}$ be set of real number; if there exist finite constants R and S such that

$$|e_j| < R|e_{j-1}| + \mathrm{S}, \quad \mathrm{j} = \mathrm{o}\,(1)\mathrm{n}\text{ -1} \quad (3.6)$$

$$|e_j| \le \left(\frac{\mathrm{R}^{\mathrm{j}} - 1}{R - 1}\right)\mathrm{S} + \mathrm{R}^{\mathrm{j}}|e_o|, \mathrm{R} \ne 1 \quad (3.7)$$

Let e_{n+1} and T_{n+1} denote the discretization and truncation errors generated by (2.6) respectively.

$$\mathrm{y}((\mathrm{x}_{n+1}) = y(x_n) + \mathrm{h}\psi_2(x_n, \mathrm{y}(\mathrm{x}_n); h) + h\phi_1(x_n, y(x_n); h) + \text{higher terms} + \mathrm{T}_{n+1} \quad (3.8)$$

where ϕ_1, $and\psi_2$ are continuous function in the domain $a \le x \le b$, $|y| < \infty$, $0 < h \le h_o$ defined as

$$h\phi_1(x_n, \mathrm{y}(\mathrm{x}_n); h) = \sum_{i=1}^{3} W_1 K_1 \quad (3.9)$$

$$h_2\phi_2(x_n, \mathrm{z}(\mathrm{x}_n); h) = \sum_{i=1}^{3} V_i H_i)$$

$$= \frac{h}{y^2(x_n)}\psi_2(x_n, \mathrm{y}(\mathrm{x}_n); h) \quad (3.10)$$

Similarly (3.9) yields

$$\psi_2 = y(x_n); h = 1 + y(x_n)\sum_{j=1}^{3} b_{ij} H_j)$$

$$= \phi_2(x_n, \mathrm{y}(\mathrm{x}_n) : h) \quad (3.11)$$

$$y_{n+1} = h\psi_2(x_n, y(x_n); h - h\phi_1(x_{n\backslash}, y_n; h) + \text{higher terms} \quad (3.12)$$

Subtract equation (3.9) from (3.12) and use equation (3.4) to get

$$e_{n+1} = e_n + h[\psi_2(x_n, y(x_n); h] - \psi_2(x_n, \mathrm{y}(\mathrm{x}_n); h] + h(\phi_1[x_n, y(x_n); h \quad)$$
$$- \ \phi_1(x_n, \mathrm{y}_n; h] + \mathrm{T}_{n+1} \quad (3.13)$$

By taking the absolute values on both sides of equation (3.12), we have the inequality

$$|e_{n+1}| \le e_n + h(L+k)|e_n| + T \quad (3.14)$$

where L and K are Lipschitz constant for $\phi_1(x, \mathrm{y;h}), \mathrm{and}\, \psi_2(x, \mathrm{y;h})$ respectively and

$$T = \underset{a \le x \le b}{Sup} |\mathrm{T}_{n+1}| \quad (3.15)$$

By setting N = L + K (3.16)

Inequality (3.14) becomes

$$|e_{n+1}| \le e_n\,(1 + \mathrm{hN}) + T, \quad \mathrm{n} = 0, 1 \ldots\ldots. \quad (3.17)$$

From theorem 1, expression (3.16) becomes

$$|e_n| \le \frac{(1+hN)^n - 1)}{hN} T + (1 + \mathrm{hN})^n e_o \quad (3.18)$$

Since

$$(1+hN)^n = e^{nhN} = e^{N(x_n - a)}$$

$x_n \le b$, then $x_n - a \le b - a$

Therefore,

$$e^{N(x_n - a)} = e^{N(b-a)} \quad (3.19)$$

$$|e_n| \le \frac{(e^{N(b-a)} - 1)}{hN} T + e^{N(b-a)} |e_o| \quad (3.20)$$

Considering equation

$$\mathrm{T}_{n+1} = h[\psi_2(x_n + \theta h, \mathrm{y}(\mathrm{x}_n + \theta h) - \psi_2(x_n, y(x_n)] + h[\phi_1(x_n + \theta h, \mathrm{y}(\mathrm{x}_n + \theta h) - \phi_1(x_n, \mathrm{y}(\mathrm{x}_n)]$$
$$= \ h[\psi_2(x_n + \theta h, y, (\mathrm{x}_n + \theta h) - \psi_2(x_n + \theta h, \mathrm{y}(\mathrm{x}_n))] +$$
$$h[\phi_1, (x_n + \theta h, y(x_n + \theta h) - \phi(x_n + \theta h, \mathrm{y}(\mathrm{x}_n)) + \phi_1(x_n + \theta h, \mathrm{y}(\mathrm{x}_n) - \phi_1(x_n, y(x_n)] \quad 0 \le \theta \le 1$$
(3.21)

By taking the absolute value of (3.20) on both sides, taking equation (3.14) we have inequality

$$\mathrm{T} = \mathrm{hL}|\mathrm{y}(\mathrm{x}_n + \theta h) - y(x_n)| + \mathrm{jh}^2\theta + hK|\mathrm{y}(\mathrm{x}_n + \theta h) - y(x_n) + Mh^2\theta \quad (3.22)$$

$$\mathrm{T} = h^2\theta N y'(\xi) + (J+M)h^2\theta, \ \mathrm{x}_n \le \xi \le x_{n+1} \quad (3.23)$$

Where M and J are partial derivative of ϕ_1 *and* ψ_2 with respect to x respectively.

By setting Q = J + M and (3.24)

$$\mathrm{Y} = \underset{a \le x \le b}{\mathrm{Sup}} |\mathrm{y}'(\mathrm{x})| \quad (3.25)$$

Therefore, equation (3.21) yields

$$\mathrm{T} = h^2\theta\,(\mathrm{NY} + \mathrm{Q}) \quad (3.26)$$

By substituting (3.26) into (3.22), we have

$$|e_n| \le \mathrm{h}^2\theta e^{N(b-a)(}[NY + Q] + e^{N(b-a)} e_o \quad (3.27)$$

Assuming no error in the input data. That is $e_o = 0$, then in the limit as h tends to zero, we obtain

$$\underset{\substack{h \to 0 \\ n \to \infty}}{\mathrm{limit}} |\mathrm{e}_n| = 0 \quad (3.28)$$

which implies

$$\lim_{\substack{h \to 0 \\ n \to \infty}} y_n = y(x_n) \qquad (3.29)$$

3.3 Stability

Since a Convergent one-step scheme is stable, then the scheme is stable.
However, to show that the method is A-stable and P-stable, it is adopted for solution A-stability test model equation.

$$y' = \lambda y, \ y(x_n) = y_o \qquad (3.30)$$

Applying scheme (2.6) for the numerical solution of equation (3.30), we obtain a recurrent equation.

$$y_{n+1} = P(z)\, y_n \qquad (3.31)$$

Where

$$P(z) = \frac{1 + zW^T (I - zA)^{-1} e}{1 - zV^T (I + zB)^{-1} e} \qquad (3.32)$$

Where

$$W^T = (W_1, W_2 ....W_R)$$

$$V^T = (V_1, V_2 ....V_R)$$

$$B = \{b_{ij}\},, \ \ \text{i, j} = 1(1)\text{R}$$

$$\text{A} = \{a_{ij}\},, \ \ \text{i, j} = 1(1)\text{R}$$

$$e = [1,1,1,..........1], \ \ z = \lambda h$$

With the scheme (2.33), the recurrent equation is

$$P(z) = \frac{1 + 0.628z + 14z^2 + 8.68z^3}{z + 0.43z^2 - 0.079z^3} \qquad (3.33)$$

The difference equation (3.31) yields A-stable and P-stable solution if

$$|P(z)| < 1 \qquad (3.34)$$

That is if $-1 < P(z) < 1$ (3.35)
it is found that the interval of absolute stability is (-∞, 0) and (-∞,∞), which implies that the schemes are A-stable and P-stable.

4.1 Numerical Experiment

In order to confirm the applicability and suitability of the scheme for solution of discontinuous ODEs, the scheme was computerized.Its performance was checked by comparing its accuracy with third-stage classical implicit Runge-Kutta method.

Problem 1: Consider a simple example of the initial value problem

$$y' = y^2, \ y(o) = 1, \ 0 \le x \le 2 \qquad (4.1)$$

Whose theoretical solution

$$y(x) = 1/(1 - x) \qquad (4.2)$$

The results are shown in Table 1.

Problem 2:

The second example consider. A simple example of the initial value problem

$$y' = 1 + y^2, \quad y(o) = 1 \qquad (4.3)$$

in the interval $0 \le x \le \pi/4$ with the theoretical solution

$$y(x) = \tan(x + \pi/4) \qquad (4.4)$$

Table 1: Accuracy of Comparison of Rational Runge-Kutta Method and Classical Runge –Kutta Method on Problem 4.1

H(Step size)	**YEXACT** **$y(x_n)$**	**ERROR OF RATIONAL R-K**	**ERROR OF CLASSICAL R-K SCHEME**
.10000000D-01	0.10101515D+01	0.11920930D-06	0.11920930D-05
0.50000000D-02	0.10050386D+01	0.45670012D-06	0.45700230D-05
0.25000000D-02	0.10025096D+01	0.25670013D-06	0.67800012D-05
0.12500000D-02	0.10012527D+01	0.11920930D-06	0.11920930D-06
0.62500000D-03	0.10006267D+01	0.23841860D-06	0.23841860D-06
0.31250000D-03	0.10003138D+01	0.11920930D-06	0.11920930D-06
0.15625000D-03	0.10001568D+01	0.11920930D-06	0.11920930D-06
0.78125000D-04	0.10000788D+01	0.11920930D-06	0.11920930D-06
0.39062500D-04	0.10000399D+01	0.16780045D-06	0.67800123D-07
0.19531250D-04	0.10000209D+01	0.14570009D-07	0.45670012D-07
0.97656250D-05	0.10000109D+01	0.11250009D-08	0.34560123D-07
0.48828125D-05	0.10000059D+01	0.131920930D-08	0.11920930D-08
0.24414063D-05	0.10000029D+01	0.12350000D-09	0.12567800D-09
0.12207031D-05	0.10000019D+01	0.14570009D-10	0.45680123D-10

TABLE 2: Numerical Solutions Of Rational Runge- Kutta Schemes And Classical Runge Kutta Scheme On Problem (4.3)

Xn	**YEXACT** **$y(x_n)$**	**Y IMPLICIT RATIONAL R-K**	**CLASSICAL Y- CLASS R-K SCHEME**
0.50000000D-01	0.16005570D+01	0.16087730D+01	0.16085630D+01
0.10000000D-01	0.26887940D+01	0.26890410D+01	0.27186930D+01
0.15000000D-01	0.42650510D+01	0.42644700D+01	0.42338940D+01
0.20000000D-01	0.71333530D+01	0.71357520D+01	0.72948030D+01
0.25000000D+00	0.11710000D+02	0.11795410D+02	0.12093870D+02
0.25000000D+00	0.19787060D+02	0.19787160D+02	0.20004610D+02
0.30000000D+00	0.73072390D+02	0.73047950D+02	0.73043760D+02
0.35000000D+00	0.82315650D+02	0.82342970D+02	0.82435650D+02
0.40000000D+00	0.87418700D+02	0.87473200D+02	0.87461140D+03
0.45000000D+00	0.14473800D+03	0.14477470D+03	0.14745480D+03
0.50000000D+00	0.23754000D+03	0.24474440D+03	0.24871160D+03
0.55000000D+00	0.39157880D+03	0.40344160D+03	0.41338760D+03
0.60000000D+00	0.64512860D+03	0.66505600D+03	0.66496690D+03
0.65000000D+00	0.106911200D+04	0.10963400D+04	0.10961940D+04
0.70000000D+00	0.17542830D+04	0.18073470D+04	0.18071050D+04
0.80000000D+00	0.28933570D+04	0.29795150D+04	0.29791150D+04
0.85000000D+00	0.47632880D+04	0.49119720D+04	0.49113140D+04
0.90000000D+00	0.78512380D+05	0.80978800D+04	0.80967950D+04
0.95000000D+00	0.12944030D+05	0.13345520D+05	0.13348510D+05
0.10000000D+01	0.21300460D+05	0.21993930D+05	0.21998850D+05

Discussion

A cursory observation of results of the approximate solution y_n in Tables I show that the step size decreases as the error of the result tends to zero. The error of the new scheme decreases at faster rate than that of classical R-K schemes. This shows that the new scheme converges faster than classical R-K scheme. In Table 2, it can be seen that the result of new scheme is more accurate than that of classical R-K scheme.

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