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Solving Directly General Fifth Order Ordinary Differential Equations by Block Predictor-Corrector method.

Olabode B. T
Department of Mathematical Sciences,
Federal University of Technology, Akure, Nigeria.

Abstract

The direct solution of fifth order initial and boundary value problems of ordinary differential equations using block predictor-corrector method was proposed in this article. The linear multistep method was developed by collocation approximation method and was then applied as simultaneous integrator in block form for the direct solution of fifth order initial value and boundary problems of ordinary differential equations in non-parallel mode. The prediction equation was obtained directly from the general block formula. Moreover, a zero stable method obtained, possessed the desirable feature of Runge-Kutta method of being self-starting and eliminated the use of predictors. The numerical results are presented to illustrate the effeciency of the method.

Keywords: Linear multistep methods (LMMs); Fifth order; Initial Value Problems (IVPs); Ordinary Differential Equations (ODEs); Interval of periodicity; Predictor-corrector (P-C).

1.0 Introduction

Consider the higher order initial value problem of ordinary differential equation of the form;

$$y^n = f(x, y(x), y'(x), y''(x), \dots, y^{n-1}(x)),$$
$$y(a) = y_0, \dots, y^i(a) = y_i, \qquad i = 1(1)n-1, n \geq 5 \qquad (1.1)$$

for step number k ≥ 5. This class of problems has a wide variety of applications in science and engineering field especially in mechanical systems without dissipation, control theory and celestial mechanics. However, only a limited number of analytical methods are available for solving (1.1) directly without reducing it to a first order system of initial value problems.

There are considerable literatures on the solution of higher order initial value problems of ordinary differential equations (see [1-5]). In particular, [6] developed a p-stable linear multistep method for solving third order ordinary differential equations. These authors respectively proposed linear multistep methods with continuous coefficients where they adopted Taylor series expansion to supply starting values. Although predictor-corrector method yielded good results but the implementation is too costly, for instance, predictor-corrector (P-C) subroutines are very complicative to write since they require special techniques to supply the starting values. This leads to longer computer time and human effort. Details can be found in [7] and [6].

In recent time, authors have adopted the block method for solving higher order ordinary differential equations (see [8] - [13]). In [14] a two-point four step direct implicit block method of order 7 for solving third order ordinary differential equations using variable step size strategy was proposed.

Definition 1.1: The block method of RXR matrix is zero-stable provided the roots R_j, $j = 1, 2, \dots k$ of the first characteristic polynomial $\rho(R)$ specified as $\rho(R) = \det\left[\sum_{i=0}^{k} A^i R^{k-i}\right] = 0$, satisfies $|R_j| \leq 1$ and for those roots with $|R_j| = 1$, the multiplicity must not exceed the order of the differential equation (see [15]).

Moreover, in reference [15], block predictor-corrector method for second order initial value problems denoted by P(EC)$^\mu$ with $\mu \geq 1$ is presented. The implementation in $P(EC)^\mu$ E-mode is described as follows:

Let Y_m^0 denote the numerical solution that will be obtained with the predictor. $Y_{m,1}$ denote the numerical solution that will be obtained with the corrector method. $Y_{m,2}$ is the numerical solution that will be obtained with the optimal corrector method.

Corresponding author : ***Olabode B. T,*** E-mail: **-**, Tel. +2348039739771

$$P: \quad Y_m^0 = \tilde{A}^{(1)}Y_{m-1}^{(\mu)} + h^2\tilde{B}^{(1)}F_{m-1}^{(\mu)} \quad (1.2)$$

$$E: \quad F_m^{(t)} = \begin{bmatrix} f\left(x_{n+1}, y_{n+1}^{(t)}\right) \\ f\left(x_{n+2}, y_{n+2}^{(t)}\right) \end{bmatrix} \quad (1.3)$$

$$C: \quad Y_m^{(t+1)} = A^{(1)}Y_{m-1}^{(\mu)} + h^2\left[B^0F_m^{(t)} + B^{(1)}F_{m-1}^{(\mu)}\right], \quad (1.4)$$

for $t = 0, 1, \dots, \mu - 1$

$$E: \quad F_m^{(\mu)} = \begin{bmatrix} f\left(x_{n+1}, y_{n+1}^{(\mu)}\right) \\ f\left(x_{n+2}, y_{n+2}^{(\mu)}\right) \end{bmatrix} \quad (1.5)$$

The effect of the P(EC)$^{\mu}$ E-mode can be observed if by making the localizing assumption that Y_{m-1} is exact; that is $W_{m-1} = Y_{m-1}$, then the error is given as:

$$W_m - Y_m^{(t+1)} = h^2B^{(0)}[W_m'' - F_m^{(t)}] + \tau_m \quad (1.6)$$

$$= h^2LB^{(0)}\left[W_m - Y_m^{(t)}\right] + \tau_m \quad (1.7)$$

where L is the Lipschitz constant of $f(x, y)$ with respect to y. Equation (1.7) suggests that the order of the predictor must be closed to that of the corrector lest it lower the accuracy of the solution. Possible estimate of the local truncation error is obtained as:

$$\text{n}\tau_m = \left\|Y_{m,2} - Y_{m,1}\right\|. \quad (1.8)$$

As a complemet to [15], reference [16] and [17] independently proposed block predictor-corrector method which is self-starting since the prediction equation is obtained directly from the general block formula. Reference [16] gave the general discrete block formula as:

$$A^{(0)}Y_m = ey_n + hdf(y_n) + h^{\mu}Bf(Y_m^0) \quad (1.9)$$

where the prediction equation is given by

$$Y_m^0 = ey_n + h^{\mu}df(y_n) \quad (1.10)$$

Putting (1. 10) into (1.9) one obtains

$$A^{(0)}Y_m = ey_n + hdf(y_n) + h^{\mu}ey_n + h^{\mu}df(y_n) \quad (1.11)$$

According to [16], (1.11) is called block predictor-corrector method and it is self-starting. The prediction equation is obtained directly from the general block formula. This paper therefore presents a new block predictor-corrector method for the direct solution of fifth-order initial and boundary value problems of ordinary differential equations in the spirit of [15] and [16].

The study is organized as follows; the second section describes the method, special application of the method and the analysis of the basic properties such as convergence, order, error constant and zero-stability. Finally, the numerical experiments and the results are presented in the third section.

2.0 Method

A power series of a single variable x in the form:

$$P(x) = \sum_{j=0}^{\infty} a_j x^j \quad (2.1)$$

is used as the basis or trial function, to produce the approximate solution as

$$y(x) = \sum_{j=0}^{k+1} a_j x^j \quad (2.2)$$

Where $a_j \in R, j = 0(1)k + 1,\ y \in C^m, (a, b) \subset P(x)$. Assuming an approximate solution to (1.1) in the form of (2.2) whose higher derivatives are

$$y'(x) = \sum_{j=0}^{k+1} ja_j x^{j-1} \quad (2.3)$$

$$y''(x) = \sum_{j=0}^{k+1} j(j-1)a_j x^{j-2} \quad (2.4)$$

$$\vdots$$

$$y^n(x) = \sum_{j=0}^{k+1} j(j-1)(j-2)\dots(j-(n-1))a_j x^{j-n} \quad (2.5)$$

From (1.1) and (2.5) one obtains

$$f(x,y,y',y'',\dots,y^{n-1})=\sum_{j=0}^{k+1} j(j-1)(j-2)\dots(j-(n-1))a_j x^{j-n} \qquad (2.6)$$

where $a_j's$ are the parameters to be determined. Collocating equation (2.6) at the mesh-points $x=x_{n+j}, j=0(1)k$ and interpolating equation (2.2) at $x=x_{n+j}, j=0(1)k-1$.

$$f_{n+j}=\sum_{j=0}^{k+1} j(j-1)(j-2)\dots(j-(n-1))a_j x^{j-n} \qquad (2.7)$$

$$y_{n+j}=\sum_{j=0}^{k+1} a_j x_{n+j,}^{j} \quad j=0(1)k-1, \qquad (2.8)$$

which after some manipulation, one obtains the new continuous method expressed of the form:

$$y_5(x)=\sum_{j=0}^{k-1}\alpha_j(x)y_{n+j}+h^5\sum_{j=0}^{k}\beta_j(x)f_{n+j} \qquad (2.9)$$

The coefficients $\alpha_j(x)$ and $\beta_j(x)$ are expressed as functions of t as follows:

$$\alpha_0(t)=\frac{1}{24}(t^4+6t^3+11t^2+6t)$$
$$\alpha_1(t)=\frac{1}{6}(t^4+7t^3+14t^2+8t)$$
$$\alpha_2(t)=\frac{1}{4}(t^4+8t^3+19t^2+12t)$$
$$\alpha_3(t)=\frac{1}{6}(t^4+9t^3+26t^2+24t)$$
$$\alpha_4(t)=\frac{1}{24}(t^4+10t^3+35t^2+50t+24)$$
$$\alpha_5(t)=1$$
$$\beta_0(t)=\frac{h^5}{725760}(-2t^{10}-20t^9-45t^8+120t^7+504t^6+4285t^5-5670t^4-820t^3+948t^2+720t)$$
$$\beta_1(t)=\frac{h^5}{1451520}(2t^{10}+24t^9+63t^8-144t^7-672t^6-4285t^5+8652t^4+16104t^3+26460t^2+14256t)$$
$$\beta_2(t)=\frac{h^5}{725760}(-2t^{10}-28t^9-99t^8+168t^7+1008t^6+4285t^5+8064t^4+84532t^3+153084t^2+81648t)$$
$$\beta_3(t)=\frac{h^5}{725760}(2t^{10}+32t^9+153t^8-48t^7-2016t^6-4285t^5+38682t^4+109168t^3+133764t^2+57168t)$$
$$\beta_4(t)=\frac{h^5}{1451520}(-2t^{10}-36t^9-225t^8-360t^7+2184t^6+16381t^5+19950t^4+20700t^3+4068t^2-2160t)$$
$$\beta_5(t)=\frac{h^5}{7257600}(2t^{10}+40t^9+315t^8+1200t^7+2016t^6-4285t^5-3400t^3+1932t^2+2160t)$$

(2.10)

Evaluating (2.10) at t = 1 which implies $x=x_{n+5}$ one obtains the discrete scheme

$$y_{n+5}-5y_{n+4}+10y_{n+3}-10y_{n+2}+5y_{n+1}-y_n=\frac{h^5}{362880}(-f_{n+5}+15125f_{n+5}+166310f_{n+3}+166330f_{n+2}+15115f_{n+1}+f_n) \qquad (2.11)$$

Equation (2.11) is consistent, of order $p=8$ with the error constant $C_{p+2}=\frac{1}{362880}=2.755731923*10^{-6}$. The region of absolute stability is (0, -0.5643).

2.1 *Special Application of the Method*

Here the special application of this work is presented. Differentiating (2.10) and evaluating at $t=0(1)5$ one obtains the following:

$$y_{n+5}-5y_{n+4}+10y_{n+3}-10y_{n+2}+5y_{n+1}-y_n=\frac{h^5}{362880}(-f_{n+5}+15125f_{n+5}+166310f_{n+3}+166330f_{n+2}+15115f_{n+1}+f_n) \qquad (2.12)$$

$$y'_{n+5}=\frac{1}{12h}(77y_{n+4}-214y_{n+3}+234y_{n+2}-122y_{n+1}+25y_n)+\frac{h^4}{161280}(-49f_{n+5}$$

$$+26469f_{n+4}+172134f_{n+3}+155834f_{n+2}+13819f_{n+1}+49f_n) \quad (2.13)$$

$$y''_{n+5}=\frac{1}{12h}(71y_{n+4}-236y_{n+3}+294y_{n+2}-164y_{n+1}+35y_n)+\frac{h^4}{453600}(1834f_{n+5}$$
$$229495f_{n+4}+780985f_{n+3}+640865f_{n+2}+45905f_{n+1}+1916f_n) \quad (2.14)$$

$$y'''_{n+5}=\frac{1}{2h}(7y_{n+4}-26y_{n+3}+36y_{n+2}-22y_{n+1}+5y_n)+\frac{h^4}{120960}(5731f_{n+5}$$
$$+139763f_{n+4}+190542f_{n+3}+177542f_{n+2}-2417f_{n+1}+2919f_n) \quad (2.15)$$

$$y^{iv}{}_{n+5}=\frac{1}{24h}(y_{n+4}-4y_{n+3}+6y_{n+2}-4y_{n+1}+y_n)+\frac{h^4}{60480}(15665f_{n+5}$$
$$+101309f_{n+4}+998f_{n+3}+77222f_{n+2}-20039f_{n+1}+4285f_n) \quad (2.16)$$

Equations (2.12)-(2.16), in block form, are implemented using a MATLAB algorithm, gives solution at selected grid points within the integration interval.

2.2 Analysis of the Method

The efficiency of any of the proposed methods depend on the stability and some accuracy properties. Some of these properties include the order, zero stability and the convergence of the method. These basic properties were investigated in this section.

2.2.1 Order of the method

The methods proposed by [1] and [15] were employed in obtaining the order of the block methods equations (2.12)-(2.16) as $[8,\ 8,\ 8,\ 8,\ 8\,]^T$ and the error constants are $[\frac{1}{362880},\frac{252589}{21772800},\frac{2589}{5443200},\ \frac{2579}{120960},\ \frac{143}{2520}]^T$.

2.2.2 Zero Stability of the block method

$$A=det\left[z\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{pmatrix}=0\right]$$

$$A=z^5-z^4=0,0,0,0,1 \quad (2.17)$$

Hence the block method is zero stable (see definition 1.1). The block method is also consistent, as it has the order p greater than 1. Hence the convergence of the method is asserted as in [18].

2.2.3 *Region of Absolute Stability of the method*

In this article, the Locus method is used to determine the region of absolute stability. The boundary locus method is given by

$$h(\theta)=\frac{\rho(r)}{\sigma(r)}=\frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} \quad (2.18)$$

where $\rho(r)$ and $\sigma(r)$ are the first and second characteristics polynomial respectively as:

$$r=e^{i\theta}=cos\theta+isin\theta \quad (2.19)$$

Applying (2.18) and (2.19) to the derived scheme, we have

$$h(\theta)=\frac{362880(r^5-5r^4+10r^3-10r^2+5r+1)}{(-r^5+15125r^4+166310r^3+166330r^2+15115r+1)} \quad (2.20)$$

$$h(\theta)=$$
$$\frac{362880(cos5\theta-5cos4\theta+10cos3\theta-10cos2\theta+5cos\theta+1)+i(sin5\theta-5sin4\theta+10sin3\theta-10sin2\theta+5cos\theta)}{(-cos5\theta+15125cos4\theta+166310cos3\theta+166330cos2\theta+15115cos\theta+1)+i(-sin5\theta+15125sin4\theta+166310cos3\theta+166330sin2\theta+15115sin\theta)}$$

Multiplying by conjugate, considering the real part and evaluating at intervals of $(0,180^0)$ one obtains (0, -0.5643).

3.0 Numerical experiments and results

This section deals with numerical experiments and results. The following initial and boundary value problems of ordinary differential equations are considered.

Problem 3.1

$$y^v=2y'y''-yy^{iv}-y'y'''-8x+(x^2-2x-3)e^x;0\le x\le 1,$$
$$y(0)=0,y'(0)=1,y''(0)=3,y'''(0)=1,y^{iv}(0)=1$$

Exact solution$y(x)=e^x+x^2$

Problem 3.2

$$y^{v} = 6(2(y')^3+6yy'y''' + y^2y''';1 \le x \le 2,$$
$$y(1) = 1, y'(1) = -1, y''(1) = 2, y'''(1) = -6, \quad y^{iv}(1) = 24$$

Exact solution$y(x) = \frac{1}{x}$

Table3.1: The y-Exact, y-Approximate and Error in the initial value problem 3.1, with h = 0.1

X	y-Exact	y-Approximate	Error in block(P-C)method
0.1	1.1151709181	1.1151709181	4.441D-16
0.2	1.2614027582	1.2614027582	1.177D-015
0.3	1.4398588076	1.4398588076	1.315D-015
0.4	1.6518246976	1.6518246976	1.712D-014
0.5	1.8987212707	1.8987212707	1.044D-014
0.6	2.1821188004	2.1821188004	8.434D-015
0.7	2.5037527075	2.5037527075	1.155D-014
0.8	2.8655409285	2.8655409285	1.199D-014
0.9	3.2696031112	3.2696031112	1.155D-014
1.0	3.7182818285	3.7182818285	1.332D-014

In table 3.1, y-exact, y-approximate and error in the initial value problem 3.1, using Block (P-C) method is shown.

Table3.2: The y-Exact, y-Approximate and Error in the boundary value problem 3.2, with h = 0.1

X	y-Exact	y-Approximate	Error in block(P-C)method
1.1	0.9090909091	0.9090909091	2.227D-016
1.2	0.8333333333	0.8333333333	5.551D-016
1.3	0.7692307692	0.7692307692	1.118D-016
1.4	0.7142857143	0.7142857143	2.225D-016
1.5	0.6666666667	0.6666666667	5.551D-016
1.6	0.6250000000	0.6250000000	4.441D-016
1.7	0.5882352941	0.5882352941	5.551D-016
1.8	0.5555555556	0.5555555556	8.882D-016
1.9	0.5263157895	0.5263157895	1.776D-015
1.0	0.5000000000	0.5000000000	2.227D-015

Table 3.2 shows the y-exact, y-approximate and error in the boundary value problem 3.2, using Block (P-C) method with h = 0.1.

Conclusion

In this paper, the direct block predictor-corrector method for solving general higher order ordinary differential equations has been proposed. This method does not require developing separate predictors to implement and it is better than the conventional predictor-corrector (P-C) method. Also, the method derived were analyzed and found to be consistent, zero-stable and convergent.

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