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Exact Solution Of The Stochastic Edwards-Wilkinson Equation With Reduced Differential Transform Method

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Abstract

A one step continuous hybrid block method derived from Second Derivative approach is constructed and used to generate Initial Value Methods (IVMs) for initial value problems of stiff ordinary differential equations. The IVMs are applied as simultaneous numerical integrators by assembling them as a single block matrix equation called Second Derivative Block Hybrid Method(SDBHM) which is A(α)-stable. Numerical results produced by the block method show that the method is competitive with existing ones in the literature.

AMS Subject Classification: 65L05, 65L06
Keywords: Initial value problem, off-step point, block hybrid method, second derivative

1.0 Introduction

Consider the initial value problem of the form

$$y' = f(x, y), \qquad y(a) = y_0, \qquad x \in [a, b] \tag{1}$$

where f satisfies a Lipschitz condition. [1].
In the literature, several authors have proposed various techniques including hybrid method for the solution of (1), [2–9] and references therein.
Hybrid method (2) is the modified form of the k-step linear multistep method (LMM) obtained by incorporating off-step points in the derivation process in order to overcome the Dahlquist barrier theorem.

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k+v} \beta_{jr} \; f_{n+jr}, \tag{2}$$

where h is the step size , k is the step number, v is the number of off-points, r is a rational number for some j, $\alpha_k = 1$,and α_j, β_{jr}, are unknown constant which must be determined. Gupta [10] noted that the design of algorithms for hybrid methods is more tedious due to the occurrence of off-step functions which increase the number of predictors needed to implement
the methods. Second derivative methods proposed by Enright [11], were shown to be of order up to k + 2 were and implemented in a variable order, variable-step mode.. In this paper we proposed a second derivative hybrid method of the form (3)

$$\sum_{j=0}^{k} \alpha_j \, y_{n+j} = h\left(\sum_{j=0}^{k} \beta_j \, f_{n+j} + \sum_{j=1}^{v} \beta_{\tau_j} f_{n+\tau_j}\right) + h^2\left(\sum_{j=0}^{k} \gamma_j \, g_{n+j} + \sum_{j=1}^{v} \gamma_{\tau_j} \, g_{n+\tau_j}\right) , \tag{3}$$

with additional methods that are combined and implemented as a block method .
The block hybrid method proposed in this paper is developed via collocation and interpolation procedure ([1], [12–14]).The main hybrid method together with additional methods is applied as a block method to simultaneously solve (1).
The continuous representation generates a main discrete SDHM and two additional method to simultaneously produce approximations $\left(y_{n+\frac{1}{5}}, \, y_{n+\frac{3}{5}} \, , \, y_{n+1}\right)$, at a block of points, $\left(x_{n+\frac{1}{5}}, \, x_{n+\frac{3}{5}} \, , \, x_{n+1}\right)$, h=$x_{n+1} - x_n$, n = 0, . . . ,N − 1 on a partition [a, b], where a, b ∈ R, h is the constant step-size, n is a grid index and N > 0 is the number of steps. The method preserves the Runge-kutta traditional advantage of being self- starting and is more accurate since it is implemented as a block method. It should be noted that block methods were first introduced by Milne [15] for the purpose of obtaining starting values for predictor-corrector algorithms (see Sarafyan [16]). However, Rosser [17], developed Milne's idea into algorithms

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for general use. Block methods have also been considered by Shampine and Watts [18]. In order to apply the proposed block method at the next block to obtain y_{n+2}, the only necessary starting value is y_{n+1}, and the loss of accuracy in y_{n+1} does not affect subsequent points, thus the order of the algorithm is maintained. It is not required to make a function evaluation at the initial part of the new block. Thus, at all blocks except the first, the first function evaluation is already available from the previous block. Hence, as we proceed we have six function evaluations per step. Inspite of the higher order of our method, the method is also very efficient.

The paper is organized as follows. In Section 2, we obtain a continuous representation Y(x) for the exact solution y(x) which is used to generates a main discrete SDHM and two additional method for solving (1). The analysis of the method is discussed in Section 3. Numerical examples are given in Section 4 to show the efficiency of the method. Finally, the conclusion is presented in Section 5.

2.0 Development of the Second Derivative Hybrid Block Method

A continuous representation of the hybrid method is derived and used to generate the main discrete method of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\left(\sum_{j=0}^{k} \beta_j f_{n+j} + \sum_{j=1}^{v} \beta_{\tau_j} f_{n+\tau_j}\right) + h^2\left(\sum_{j=0}^{k} \gamma_j g_{n+j} + \sum_{j=1}^{v} \gamma_{\tau_j} g_{n+\tau_j}\right) \qquad (4)$$

where h is the step size, $v = 2$ is the number of off-step points, $\alpha_k = 1$, α_j, β_j, β_{τ_j} are unknown constants and τ_j are non-negative rational numbers.

In order to obtain (4), we assume a continuous solution $Y(x)$ of the form (5) as an approximation of the exact solution $y(x)$

$$Y(x) = \sum_{j=0}^{p+2q-1} a_j \varphi_j(x) \qquad (5)$$

where $x \in [x_0, x_n]$, a_j are unknown coefficients to be determined and $\varphi_j(x)$ are polynomial basis function of degree $p + 2q - 1$ such that p is the number of interpolation and the number of the collocation points $2q$ are respectively chosen to satisfy $0 \le p \le k$, $q > 0$. The proposed method is constructed by specifying the following parameters: $\tau_j = (1/5, 3/5)$, $\varphi_j(x) = x^j, j = 0, \dots, 8$, $p = 1, q = 4$ and $k = 1$, with the assumption that $y_{n+\frac{1}{5}}$ denote numerical solution of the exact solution $y\left(x_{n+\frac{1}{5}}\right)$, $f_{n+\frac{1}{3}} = f\left(x_{n+\frac{1}{3}}\right)$ and $g_{n+\frac{1}{3}} = g\left(x_{n+\frac{1}{3}}\right)$, n is the grid index.

Letting $\varphi_j(x) = x^j$, $j = 0,1, \dots, p + 2q - 1$, we impose that the interpolating function (5) coincides with the analytical solution at the point $x_{n+i}, i = 0$ to obtain the equation

$$Y(x_{n+i}) = y_{n+i}, \ \ i = 0 \qquad (6)$$

If the function (5) satisfies the differential equation (1) at the points $x_{n+i}, i = 0, \frac{1}{5}, \frac{3}{5}, 1$, we obtain the following set of four equations:

$$Y'(x_{n+i}) = f_{n+i}, \ \ i = 0, \frac{1}{5}, \frac{3}{5}, 1 \qquad (7)$$

We further demand that the second derivative of the function (5) coincides with the second derivative of the analytical solution at the points $x_{n+i}, i = 0, \frac{1}{5}, \frac{3}{5}, 1$ to obtain the following set of four equations:

$$Y''(x_n) = g_{n+i}, \ \ i = 0, \frac{1}{5}, \frac{3}{5}, 1 \qquad (8)$$

Equations (6), (7) and (8) lead to a system of nine equations which is solved to obtain the coefficient a_j. The one-step continuous hybrid method is obtained by substituting these values of $a_j, j = 0(1)8$ into (5). After some algebraic computation, the hybrid method yields the expression in the form

$$Y(x) = y_n + h\left(\sum_{j=0}^{1} \beta_j (x) f_{n+j} + \sum_{j=1}^{2} \beta_{\tau_j}(x) f_{n+\tau_j}\right) + h^2\left(\sum_{j=0}^{1} \gamma_j (x) g_{n+j} + \sum_{j=1}^{2} \gamma_{\tau_j}(x) g_{n+\tau_j}\right) \qquad (9)$$

where $\beta_j(x)$, $\beta_{\tau_j}(x), \gamma_j(x)$ and $\gamma_{\tau_j}(x)$ are continuous coefficients. Equation (2.6) is then used to generate the main discrete hybrid method by evaluating at the point $x = x_{n+1}$ to have

$$y_{n+1} = y_n + h\left(\frac{593}{2268}f_n + \frac{5125}{21504}f_{n+\frac{1}{5}} + \frac{12625}{36288}f_{n+\frac{3}{5}} + \frac{3275}{21504}f_{n+1}\right) + h^2\left(\frac{19}{1512}g_n + \frac{575}{10752}g_{n+\frac{1}{5}} + \frac{775}{24192}g_{n+\frac{3}{5}} - \frac{73}{10752}g_{n+1}\right) \quad (10)$$

the additional methods are obtained by evaluating (2.6) at points $\{ x = x_{n+\frac{1}{5}}, x_{n+\frac{3}{5}}\}$ to obtained

$$y_{n+\frac{1}{5}} = y_n + h\left(\frac{599749}{7087500}f_n + \frac{60541}{537600}f_{n+\frac{1}{5}} + \frac{2281}{907200}f_{n+\frac{3}{5}} + \frac{16903}{67200000}f_{n+1}\right) + h^2\left(\frac{10223}{4725000}g_n - \frac{7997}{1344000}g_{n+\frac{1}{5}} - \frac{1429}{3024000}g_{n+\frac{3}{5}} - \frac{797}{33600000}g_{n+1}\right) \quad (11)$$

$$y_{n+\frac{3}{5}} = y_n + h\left(\frac{12597}{87500}f_n + \frac{47871}{179200}f_{n+\frac{1}{5}} + \frac{2073}{11200}f_{n+\frac{3}{5}} + \frac{85293}{22400000}f_{n+1}\right) + h^2\left(\frac{957}{175000}g_n + \frac{9153}{448000}g_{n+\frac{1}{5}} - \frac{1551}{112000}g_{n+\frac{3}{5}} - \frac{3807}{11200000}g_{n+1}\right) \quad (12)$$

The block hybrid method is implemented by combining and simultaneously applying the hybrid methods (10), (11) and (12) as a single block method to provide approximate solution $y_{n+\frac{1}{5}}, y_{n+\frac{3}{5}}, y_{n+1}$ for solution (1.1) at discrete block points $x = x_{n+\frac{1}{5}},\ x_{n+\frac{3}{5}}, x_{n+1},\ \ n = 0,1, \dots, N-1$ on a partition $[x_0,\ x_n]$.

3.0 Analysis of the Derived Method

The Local Truncation Error

. Following Fatunla [19], the local truncation error associated with the derived method can be defined to be the linear difference operator

$$L[y(x);h] = \sum_{j=0}^{k}\alpha_j\, y(x+jh) - h\sum_{j=0}^{k}\beta_j\, y'(x+jh) - h\sum_{j=1}^{v}\beta_{\tau_j}\, y'\big(x+\tau_j h\big) - h^2\sum_{j=0}^{k}\gamma_j\, y''(x+jh) \ - h^2\sum_{j=1}^{v}\gamma_{\tau_j}\, y''(x + \tau_j h \quad (13)$$

Assuming that $y(x)$ is sufficiently differentiable, we can expand the terms in (13) as a Taylor series about the point t to obtain the expression

$$L[y(x);h] = C_0 y(x) + hC_1 y'(x) + h^2 C_2 y''(x) + \cdots + h^m C_m y^m(x) + \cdots$$

Where the constant $C_m, m = 0,1,2, \ldots$ are given as follows:

$$C_0 = \sum_{j=0}^{k}\alpha_j$$

$$C_1 = \sum_{j=0}^{k} j\ \alpha_j - \sum_{j=0}^{k}\beta_j - \sum_{j=1}^{v}\beta_{\tau_j}$$

$$C_2 = \frac{1}{2!}\sum_{j=0}^{k} j^2\alpha_j - \sum_{j=0}^{k} j\beta_j - \sum_{j=1}^{v}\tau_j\beta_{\tau_j} - \sum_{j=0}^{k}\ \gamma_j - \sum_{j=1}^{v}\ \gamma_{\tau_j}$$

$$C_m = \frac{1}{m!}\left[\sum_{j=0}^{k} j^m\alpha_j - m\left(\sum_{j=0}^{k} j^{m-1}\beta_j - \sum_{j=1}^{v}\tau_j^{m-1}\beta_{\tau_j}\right) - m(m\ -1)\left(\sum_{j=0}^{k} j^{m-2}\ \gamma_j - \sum_{j=1}^{v}\tau_j^{m-2}\ \gamma_{\tau_j}\right)\right]$$

The newly derived method is said to have a maximal order of accuracy m if

$$C_0 = C_1 = C_2 = \cdots = C_m = 0, \qquad C_{m+1} \neq 0$$

The constant C_{m+1} is the error constant and $C_{m+1}h^{(m+1)}y^{(m+1)}(x_n)$ is the principal local truncation error at the point x_n.

Thus, the local truncation error (LTE) of the method of order m can be written as

$$LTE = C_{m+1}h^{(m+1)}y^{(m+1)}(x_n) + O(h^{m+2})$$

The values of the error constants calculated for the SDHBM (10), (11) and (12) are given as:

$$C_7 = \left[\frac{83}{31752000000}, \frac{639}{1225000000000}, \frac{5423}{99225000000000}\right]^T$$

with order $(8,\ 8,\ 8)^T$ where T is the transpose.

3.1 Zero Stability

The methods (10), (11) and (12) are combined and represented as a matrix finite difference equation in block form given by

$$A^{(1)}Y_\omega = A^{(0)}Y_{\omega-1} + h\left(B^{(1)}F_\omega + B^{(0)}F_{\omega-1}\right) + h^2\left(D^{(1)}G_\omega + D^{(0)}G_{\omega-1}\right) \qquad (14)$$

Where

$$Y_\omega = \left(y_{n+\frac{1}{5}},\ y_{n+\frac{3}{5}},\ y_{n+1}\right)^T, Y_{\omega-1} = \left(y_{n-\frac{3}{5}}, y_{n-\frac{1}{5}},\ y_n\right)^T,$$

$$F_\omega = \left(f_{n+\frac{1}{5}}, f_{n+\frac{3}{5}},\ f_{n+1}\right)^T, F_{\omega-1} = \left(f_{n-\frac{3}{5}}, f_{n-\frac{1}{5}},\ f_n\right)^T,$$

$$G_\omega = \left(g_{n+\frac{1}{5}}, g_{n+\frac{3}{5}}, g_{n+1}\right)^T, G_{\omega-1} = \left(g_{n-\frac{3}{5}}, g_{n-\frac{1}{5}},\ g_n\right)^T,$$

And h is the step length, $A^{(1)}, A^{(0)}, B^{(1)}, B^{(0)}, D^{(1)}\ and\ D^{(0)}$ are 3 by 3 matrices with real entries written as

$$A^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\quad A^{(0)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B^{(1)} = \begin{pmatrix} \frac{60541}{537600} & \frac{2281}{907200} & \frac{16903}{67200000} \\ \frac{47871}{179200} & \frac{2073}{11200} & \frac{85293}{22400000} \\ \frac{5125}{21504} & \frac{12625}{36288} & \frac{3275}{21504} \end{pmatrix},\quad B^{(0)} = \begin{pmatrix} 0 & 0 & \frac{599749}{7087500} \\ 0 & 0 & \frac{12597}{87500} \\ 0 & 0 & \frac{593}{2268} \end{pmatrix},$$

$$D^{(1)} = \begin{pmatrix} -\frac{7997}{1344000} & -\frac{1429}{3024000} & -\frac{797}{33600000} \\ \frac{9153}{448000} & -\frac{1551}{112000} & -\frac{3807}{11200000} \\ \frac{575}{10752} & \frac{775}{24192} & -\frac{73}{10752} \end{pmatrix},\quad D^{(0)} = \begin{pmatrix} 0 & 0 & \frac{10223}{4725000} \\ 0 & 0 & \frac{957}{175000} \\ 0 & 0 & \frac{19}{1512} \end{pmatrix}$$

The zero stability of the block method is concerned with all the roots of the first characteristic polynomial $\rho(R)$ of the method such that $|R| \le 1$. The root of modulus one is simple.

$$\rho(R) = \det\left[RA^{(1)} - A^{(0)}\right] = R^2(R-1)$$

The block method (14) is zero-stable for $\rho(R) = 0$ and satisfies $|R_j| \le 1$, and for those roots with $|R_j| = 1$ the multiplicity is simple thus the block method is zero-stable.

3.2 Linear Stability

The stability property of (14) is shown by applying it to the scalar test equation

$$y' = \lambda y, \quad y'' = \lambda^2 y,$$

where λ a complex number with $\mathrm{Re}(\lambda) < 0$, and yields the equation

$y_{n+1} = M(z)y_n,\ \ z = h\lambda$

where $M(z)$ is the amplification matrix given as.

$$M(z) = \left(A^{(1)} - zB^{(1)} - z^2D^{(1)}\right)^{-1}\left(A^{(0)} + zB^{(0)} + z^2D^{(0)}\right)$$

The matrix $M(z)$ has eigenvalues $(\mu_1, \mu_2, \mu_3) = (0, 0, \mu_3)$ where the dominant eigenvalue. μ_3 is a rational function of z given by

$$\mu_3(z) = \frac{1+0.55z+0.140714z^2+0.0218095z^3+0.00221333z^4+0.000144762z^5+5.07937\times10^{-6}z^6}{1-0.45z+0.0907143z^2-0.0105714z^3+0.000760952z^4-0.0000328571z^5+7.14286\times10^{-7}z^6} \qquad (15)$$

using the equations (15), the stability region is drawn and shown in Figure 1.

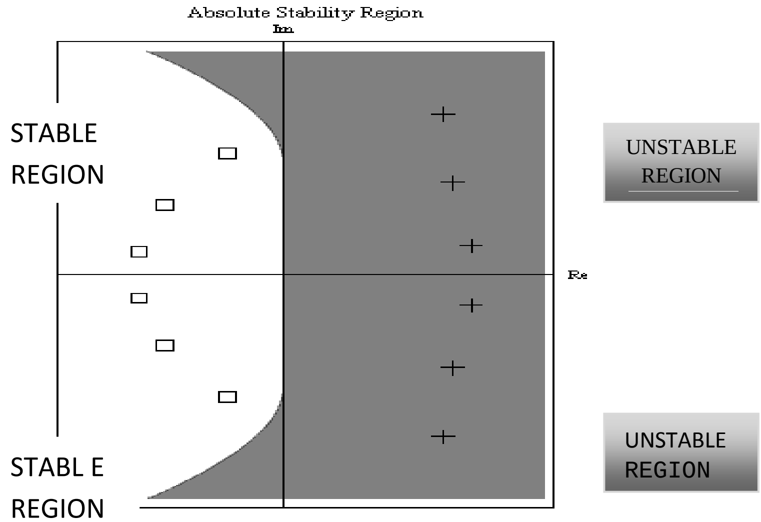

Figure 1 Stability Curve

It is obvious from figure 1 that the method is not A-stable since part of the shaded portion of the right half plane lies in the negative half plane. Thus it is A(α)-stable with α = 2.4760.

4.0 Numerical Examples

In this section, we give numerical examples to illustrate the accuracy of the derived block hybrid method. All absolute errors of the approximate solution on the partition are denotedas $|y - y(x)|$

All computations were carried out using a written code in Matlab 7.0.

Example 4.1 Our second test example is the given nonlinear IVP which was also solved by Norsett [16] and Jain[12]

$$y' = -100x\, y^2 \quad y(1) = \frac{1}{51} \qquad 0 \le x \le 20$$

Exact solution: $y(x) = \frac{1}{(1+50x^2)}$

Although the new block method (10) is not A-stable but it performed excellently than the methods given in [20] and [21] that are A-stable, when tested on this nonlinear problem. We note that though our method is expected to perform better, because of its higher order, it is observed that even for a large step size of $h = \frac{1}{4}$, the SDBHM performs better than the methods in [20] and [21] with a smaller step size of $h = \frac{1}{16}$.. Details of the numerical results are given in Table 1.

Table 1. comparison of results for Example 4.1

h	x	SDBHM (y) (| Error |)	Norsett [20] (y) (| Error |)	Jain [21] (y) (| Error |)	Exact y(x)
$\frac{1}{16}$	10	$0.19996001 \times 10^{-3}$ (1.275×10^{-15})	$0.19995554 \times 10^{-3}$ (4.470×10^{-9})	$0.19996018 \times 10^{-3}$ (1.700×10^{-9})	$0.19996001 \times 10^{-3}$
$\frac{1}{8}$	10	$0.19996001 \times 10^{-3}$ (2.753×10^{-15})	$0.19991486 \times 10^{-3}$ (4.515×10^{-8})	$0.19996310 \times 10^{-3}$ (3.090×10^{-9})	$0.19996001 \times 10^{-3}$
	20	$0.49997500 \times 10^{-4}$ (3.385×10^{-15})	$0.49994562 \times 10^{-4}$ (2.938×10^{-9})	$0.49997695 \times 10^{-4}$ (1.950×10^{-10})	$0.49997500 \times 10^{-4}$
$\frac{1}{4}$	10	$0.19996001 \times 10^{-3}$ (4.702×10^{-14})	$0.19906134 \times 10^{-3}$ (8.987×10^{-7})	$0.20000938 \times 10^{-3}$ (4.937×10^{-8})	$0.19996001 \times 10^{-3}$
	20	$0.49997500 \times 10^{-4}$ (1.389×10^{-14})	$0.49940176 \times 10^{-4}$ (5.732×10^{-9})	$0.50000607 \times 10^{-4}$ (3.107×10^{-9})	$0.49997500 \times 10^{-4}$

Example 4.2 Consider the Initial Value Problem on the range $0 \le x \le 1$,

$$y' = -y + 95z, \quad y(0) = 1$$
$$z' = -y - 97z, \quad y(0) = 1$$

Exact solution: $y(x) = \frac{95}{47}e^{-2x} - \frac{48}{47}e^{-96x}, \quad z(x) = \frac{48}{47}e^{-96x} - \frac{1}{47}e^{-2x}.$

For this problem we present and compare our result with the exact solution at the end point in Table 2.

Table 2 . Result for Example 4.2 at the end point. x=1 using SDBHM

Step	y(1) ($\|error\|$)	z(1)$\times 10^2$ ($\|error\|$) $\times 10^2$
0.125	0.27355004 (9×10^{-13})	-0.28794741 (1×10^{-10})
0.0625	0.27355004 (7×10^{-16})	0.28794741 (6×10^{-16})
0.03125	0.27355004 (3×10^{-15})	0.28794741 (3×10^{-15})

True Solution y(1) = 0.27355004 z(1) = -0.28794741

5.0 Conclusion

A one step second derivative hybrid methods have been presented and assembled into a single block matrix equation which is A(α) stable and used to simultaneously generate the solution stiff problem. In particular, the method is implemented without the need for starting values or predictors, therefore complicated subroutines are avoided. We have demonstrated the accuracy of the method on both linear and non linear problems. The numerical results given in Tables1,2 show that our method is efficient for solving stiff problems.

References

[1] P. Henrici, *Discrete Variable Methods in ODEs*, John Wiley, New York (1962).

[2] O,A. Akinfenwwa, S.N. Jator and Y.N. Yao, A Linear Multistep Hybrid methods with Continuous Coefficients for Solving Stiff Ordinary Differential Equation, journal of Morden Mathematics and Statistics 5(2)(2011)47-53

[3] L. Brugnano, D. Trigiante, Solving *Differential Problems by Multitep Initial and Boundary Value Methods*, Gordon and Breach Science Publishers,Amsterdam (1998).

[4] J.C. Butcher, A modified multistep method for the numerical integration of ordinary differential equations, J. Assoc. Comput. Mach., 12 (1965),124-135.

[5] J.R. Cash, On the exponential fitting of composite multiderivative linear multistep methods, SIAM J. Numer. Anal., 18 (1981), 808-821.

[6] C.W. Gear, Hybrid methods for initial value problems in ordinary differential equations, SIAM J. Numer. Anal., 2 (1965), 69-86.

[7] L.W. Jackson, S.K. Kenue, A fourth order exponentially fitted method, SIAM J. Numer. Anal., 11 (1974), 965-978.

[8] J.J. Kohfeld, G.T. Thompson, Multistep methods with modified predictors and correctors, J. Assoc. Comput. Mach., 14 (1967), 155-166.

[9] J.D. Lambert, *Numerical Methods for Ordinary Differential Systems*, John Wiley, New York (1991).

[10] G.K. Gupta, Implementing second-derivative multistep methods using Nordsieck polynomial representation, Math. Comp., 32 (1978), 13-18.

[11] W.H. Enright, Second derivative multistep methods for stiff ordinary differential equations, SIAM J. Numer. Anal., 11 (1974), 321-331.

[12] I. Gladwell, D.K. Sayers, Eds. *Computational Techniques for Ordinary Differential Equations*, Academic Press, New York (1976).

[13] P. Onumanyi, D.O. Awoyemi, S.N. Jator, U.W. Sirisena, New linear mutlistep methods with continuous coefficients for first order initial value problems, J. Nig. Math. Soc., 13 (1994), 37-51.

[14] I. Lie, S.P. Norsett, Superconvergence for Multistep Collocation, Math. Comp., 52 (1989), 65-79.

[15] W.E. Milne, *Numerical Solution of Differential Equations*, John Wiley andSons (1953).

[16] D. Sarafyan, Multistep methods for the numerical solution of ordinary differential equations made self-starting, Tech. Report, 495, Math. Res. Center, Madison (1965).

[17] J.D. Rosser, A Runge-kutta for all seasons, SIAM, Rev., 9 (1967), 417-452.

[18] L.F. Shampine, H.A. Watts, Block implicit one-step methods, Math. Comp., 23 (1969), 731-740.

[19] S.O. Fatunla, Numerical integrators for stiff and highly oscillatory differential equations, Math. Comp., 34 (1980), 373-390.

[20] S.P. Norsett, An A-stable modification of the Adams-Bashforth methods, In: Conference on the Numerical Solution of Differential Equations,Dundee, Scotland (1969).

[21] R.K. Jain, Some A-stable methods for stiff ordinary differential equations, Math. Comp., 26 (1972), 71-77.