Journal of the Nigerian Association of Mathematical Physics
***Volume* 23 (March, 2013), pp 109 – 112**

An Application of Block Finite Difference Method for Boundary Value Problems of Third Order Ordinary Differential Equations

[1]S. O. Soladoye and [2]Y. A. Yahaya
[1]Department of Mathematics /Computer Science and Statistics,
Kaduna Polytechnic Kaduna, Nigeria
[2]Department of Mathematics and Computer Science,
Federal University of technology Minna, Nigeria

Abstract

We propose a symmetric hybrid finite difference scheme with continuous coefficients for the solution of boundary value problems of both special and general third order ordinary differential equations. The three member block schemes of the Central, Forward and Backward difference methods derived were used simultaneously for the solution of boundary value problems of ordinary differential equations. Two numerical experiments were demonstrated to ascertain the efficiency of the proposed method.

Keywords: Symmetric, hybrid, finite difference method, Continuous coefficients, Boundary value problems

1.0 Introduction

Yahaya and Onumanyi [1] proposed a symmetric hybrid finite difference scheme with Continuous coefficients for the solution of boundary value problems of the general second order ODEs of the form

$$y'' = f(x,y,y'), \quad y(a) = y_0, \; y'(b) = y_N \quad a \subset [a,b] \qquad (1)$$

This idea is extended in this paper for the solution of boundary value problems of both special and general third order ODEs of the form

$$y''' = f(x,y)$$

$$y''' = f(x,y,y', y''), \quad y(a) = y_0, \; y'(a) = \beta, y''(a) = \mu \qquad (2)$$

Recently, some Scholars [2-4] have proposed some methods of Numerical solution of higher order Boundary value problems of ordinary differential equations. However all these methods were limited to Special and General second order boundary value problems in ordinary differential equations.

2.0 Methodology

We consider an approximate solution of the form

$$y(x) = \sum_{j=0}^{k} a_j Q_j(x) \qquad (3)$$

where $Q_j(x)$ are Canonical polynomials which is used as a basis for this method. Specifically in this method, $k = 6$.
To generate $Q_j(x)$ we write (3) in the form $y''' + y(x) = y(x) + f(x,y,y',y'')$ and define a differential operator

$$L^* = \frac{d^3}{dx^3} + 1 \qquad (4)$$

We define the Canonical polynomials $Q_j(x)$ by

$$L^* Q_j(x) = x^j \quad j = (0,1, \ldots . k) \qquad (5)$$

And we generate the Canonical polynomials $Q_j(x)$ by use of the operator L, ie

$$L^* x^j = j(j-1)(j-2)x^{j-3} + x^j \qquad (6)$$

Using equations (5) and (6) we obtain

$$L^* x^j = L^*\left[j(j-1)(j-2)Q_{j-3}(x) + Q_j(x)\right] \qquad (7)$$

Assuming L^{*-1}exist, then we have

$$Q_j(x) = x^j - j(j-1)(j-2)Q_{j-3}(x), \; j = 0,1, \ldots k \qquad (8)$$

Corresponding author: ***S.O. Soladoye***, E-mail: sikirusoladoye@yahoo.com-, Tel. +2348037850404

Thus from equation (8), we obtain recursively for $j = 0,1, \ldots .k$ where
$Q_0(x) = 1,\ Q_1(x) = x,\ Q_2(x) = x^2, Q_3(x) = x^3 - 6\ , Q_4(x) = x^4 - 24x\ ,$
$Q_5(x) = x^5 - 60x^2,\ \ Q_6(x) = x^6 - 120x^3 + 720.$
Substituting these in equation (3) yields

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3(x^3 - 6) + a_4(x^4 - 24x) + a_5(x^5 - 60x^2) + a_6(x^6 - 120x^3 + 720) \qquad (9)$$

where $a_j, j = 0,1,2,3,4,5\ and\ 6$ are parameters to be determined. We interpolate (9) at $x = x_m$ where m= $i - \frac{3}{2}, i - 1, i - \frac{1}{2}, i, i + \frac{1}{2}, i + i, i + \frac{3}{2}$, which gives the system of non linear equations of the form

$$y(x_m) = \sum_{j=0}^{6} a_j Q_j(x_m)$$

By using Maple Mathematical software, we obtain our Continuous formulation of the form

$$y(x) = A(x)y_{i-\frac{3}{2}} + B(x)y_{i-1} + C(x)y_{i-\frac{1}{2}} + D(x)y_i + E(x)y_{i+\frac{1}{2}} + F(x)y_{i+1} + G(x)y_{i+\frac{3}{2}} \qquad (10)$$

where

$$A(x) = \frac{[8(x-x_i)^6 - 12h(x-x_i)^5 - 10h^2(x-x_i)^4 + 15h^3(x-x_i)^3 + 2h^4(x-x_i)^2 - 3h^5(x-x_i)]}{90h^6}$$

$$B(x) = \frac{[-48(x-x_i)^6 + 48h(x-x_i)^5 + 120h^2(x-x_i)^4 - 120h^3(x-x_i)^3 - 27h^4(x-x_i)^2 + 27h^5(x-x_i)]}{90h^6}$$

$$C(x) = \frac{[120(x-x_i)^6 - 60h(x-x_i)^5 - 390h^2(x-x_i)^4 + 195h^3(x-x_i)^3 + 270h^4(x-x_i)^2 - 135h^5(x-x_i)]}{90h^6}$$

$$D(x) = \frac{[-160(x-x_i)^6 + 560h^2(x-x_i)^4 - 490h^4(x-x_i)^2 + 90h^6]}{90h^6}$$

$$E(x) = \frac{[120(x-x_i)^6 + 60h(x-x_i)^5 - 390h^2(x-x_i)^4 - 195h^3(x-x_i)^3 + 270h^4(x-x_i)^2 + 135h^5(x-x_i)]}{90h^6}$$

$$F(x) = \frac{[-48(x-x_i)^6 - 48h(x-x_i)^5 + 120h^2(x-x_i)^4 + 120h^3(x-x_i)^3 - 27h^4(x-x_i)^2 - 27h^5(x-x_i)]}{90h^6}$$

$$G(x) = \frac{[8(x-x_i)^6 + 12h(x-x_i)^5 - 10h^2(x-x_i)^4 - 15h^3(x-x_i)^3 + 2h^4(x-x_i)^2 + 3h^5(x-x_i)]}{90h^6} \qquad (11)$$

Taking First, Second and Third derivatives of (11) independently and substituting them in (10) when evaluated at $x = x_i$ yield First, Second and Third order derivatives Central difference schemes of the form

$$y'(x) = \frac{1}{30h}\left[-y_{i-\frac{3}{2}} + 9y_{i-1} - 45y_{i-\frac{1}{2}} + 45y_{i+\frac{1}{2}} - 9y_{i-1} + y_{i+\frac{3}{2}}\right]$$

$$y''(x) = \frac{1}{45h^2}\left[2y_{i-\frac{3}{2}} - 27y_{i-1} + 270y_{i-\frac{1}{2}} - 490y_i + 270y_{i+\frac{1}{2}} - 27y_{i-1} + 2y_{i+\frac{3}{2}}\right]$$

$$y'''(x) = \frac{1}{h^3}\left[y_{i-\frac{3}{2}} - 8y_{i-1} + 13y_{i+\frac{1}{2}} + 8y_{i+1} - y_{i+\frac{3}{2}}\right] \qquad (12)$$

Equation (12) is of order $[6,6,4]^T$ with Error constants $\left[\frac{1}{8960}\ , \frac{1}{35840}, \frac{7}{1920}\right]^T$ respectively.

Also in the same manner, evaluating (10) at $x = x_{i-\frac{3}{2}}$ and choosing $i = i + \frac{3}{2}$ gives the first, second and third derivatives Forward difference schemes of the form

$$y'(x) = \frac{1}{30h}\left[-147y_i + 360y_{i+\frac{1}{2}} - 450y_{i+1} - 450y_{i+1} + 400y_{i+\frac{3}{2}} - 225y_{i+2} + 72y_{i+\frac{5}{2}} - 10y_{i+3}\right]$$

$$y''(x) = \frac{1}{45h^2}\left[812y_i - 3132y_{i+\frac{1}{2}} + 5265y_{i+1} - 5080y_{i+\frac{3}{2}} + 2970y_{i+2} - 972y_{i+\frac{5}{2}} + 137y_{i+3}\right]$$

$$y'''(x) = \frac{1}{h^3}\left[-49y_i + 232y_{i+\frac{1}{2}} - 461y_{i+1} + 496y_{i+\frac{3}{2}} - 307y_{i+2} + 104y_{i+\frac{5}{2}} - 15y_{i+3}\right] \qquad (13)$$

Equation (13) is of Order $[6,5,4]^T$ with Error constants $\left[\frac{1}{448}\ , \frac{7}{320}, \frac{79}{240}\right]^T$ respectively.

Also evaluating (10) at $x = x_{i+\frac{3}{2}}$ and $i = i - \frac{3}{2}$ gives the first, second and third order derivatives Backward difference schemes of the form

$$y'(x) = \frac{1}{30h}\left[107y_{i-3} - 72y_{i-\frac{5}{2}} + 225y_{i-2} - 400y_{i-\frac{3}{2}} + 450y_{i-1} - 360y_{i-\frac{1}{2}} + 147y_i\right]$$
$$y''(x) = \frac{1}{45h^2}\left[137y_{i-3} - 972y_{i-\frac{5}{2}} + 2970y_{i-2} - 5080y_{i-\frac{3}{2}} + 5265y_{i-1} - 3132y_{i-\frac{1}{2}} + 812y_i\right]$$
$$y'''(x) = \frac{1}{h^3}\left[15y_{i-3} - 104y_{i-\frac{5}{2}} + 307y_{i-2} - 496y_{i-\frac{3}{2}} + 461y_{i-1} - 232y_{i-\frac{1}{2}} + 49y_i\right] \quad (14)$$

Equation (14) is of Order $[6,5,4]^T$ with Error constants $\left[\frac{1}{448}, \frac{7}{320}, \frac{29}{240}\right]^T$ respectively.

3.0 Implementation Strategies

We combine first, second and third derivatives of Central, Forward and Backward difference schemes of equations (12), (13) and (14) to form our proposed block method in solving problems of form (2). The result is obtained in block form which speed up the computational processes over non-overlapping intervals.

4.0 Numerical Experiments

The following problems were used to demonstrate the efficiency and accuracy of the proposed method.

Example 1

$y''' + sinx = 0$, $\quad y(0) = 4, y(1) = 7, y(2) = 0, \; h = 0.2$

Theoretical Solution:

$$y(x) = 5 - cosx + \frac{1}{2}(-10 - 2\cos(1) + 2\cos(1)^2)x^2 + (-\cos(1)^2 + 7 + 2\cos(1))x$$

Example 2

$y''' - y'' + y' - y = 0$, $\quad y(0) = 1, \quad y\left(\frac{\pi}{2}\right) = 0, \quad y(\pi) = 1, \quad h = (0.1)\pi$

Theoretical Solution

$y(x) = cosx$

Table 1: APPROXIMATE SOLUTION AND ABSOLUTE ERROR OF EXAMPLE 1

x	THEORETICAL SOLUTION	SFD METHOD	NEW BLOCK FDM METHOD	ERROR OF SFDM	ERROR OF NEW BFDM
0.2	5.367734	5.368614234	5.367733737	8.80234 E(-4)	2.63 E(-7)
0.4	6.354670102	6.356001349	6.354669784	1.331247 E(-3)	3.18 E(-7)
0.6	6.958455942	6.959786351	6.958455666	1.330409 E(-3)	2.76 E(-7)
0.8	7.17527525	7.176169237	7.175275094	8.93987 E(-4)	1.56 E(-7)
1.0	7.000000000	7.000000000	7.000000000	-----------------	-------------
1.2	6.426394837	6.425506759	6.426395004	8.88078 E(-4)	1.67 E(-7)
1.4	5.447365679	5.446067637	5.447365970	2.298042 E(-3)	2.91 E(-7)
1.6	4.055242512	4.053982634	4.055242851	1.259878 E(-3)	3.39 E(-7)
1.8	2.242085194	2.241276756	2.242085468	8.08438 E(-4)	2.74 E(-7)
2.0	0.0000000000	0.0000000000	0.000000000	-----------------	-----------

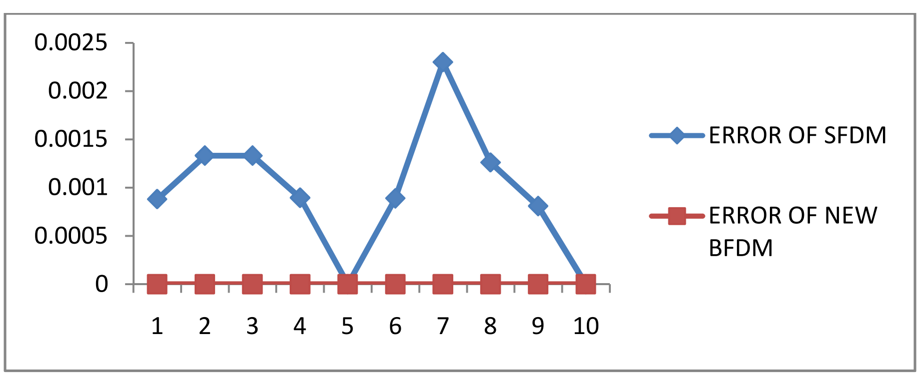

Figure 1: Error graph of Example 1
SFDM: Standard finite Difference Method BFDM: Block finite Difference Method

Table 2: APPROXIMATE SOLUTION AND ABSOLUTE ERROR OF EXAMPLE 2

x	THEORETICAL SOLUTION	SFD METHOD	NEW BLOCK FDM METHOD	ERROR OF SFDM	ERROR OF NEW BFDM
$(0.1)\pi$	0.951043927	0.9489364893	0.9512854731	2.1074 E(-3)	2.415 E(-4)
$(0.2)\pi$	0.808969105	0.8062760844	0.8093064741	2.693 E(-3)	3.3737 E(-4)
$(0.3)\pi$	0.587686382	0.5854995753	0.5880143635	2.187 E(-3)	3.2798 E(-4)
$(0.4)\pi$	0.308862026	0.3077802791	0.3091285852	1.082 E(-3)	2.666559 E(-4)
$(0.5)\pi$	0.000000000	0.00000000	0.0000000000	------------	-------------------
$(0.6)\pi$	-0.30924943	-0.3081732377	-3.3090757874	1.0762 E(-3)	1.736426 E(-4)
$(0.7)\pi$	-0.588015913	-0.58701363304	-0.5878321615	1.0022 E(-3)	1.837515 E(-4)
$(0.8)\pi$	-0.809208497	-0.8089533407	-0.8090013820	2.552 E(-4)	2.07115 E(-4)
$(0.9)\pi$	-0.951169741	-0.9516256098	-0.9509915632	4.559 E(-4)	1.781778 E(-4)
π	-1.00000000	-1.00000000	-1.0000000000	--------------	-----------------

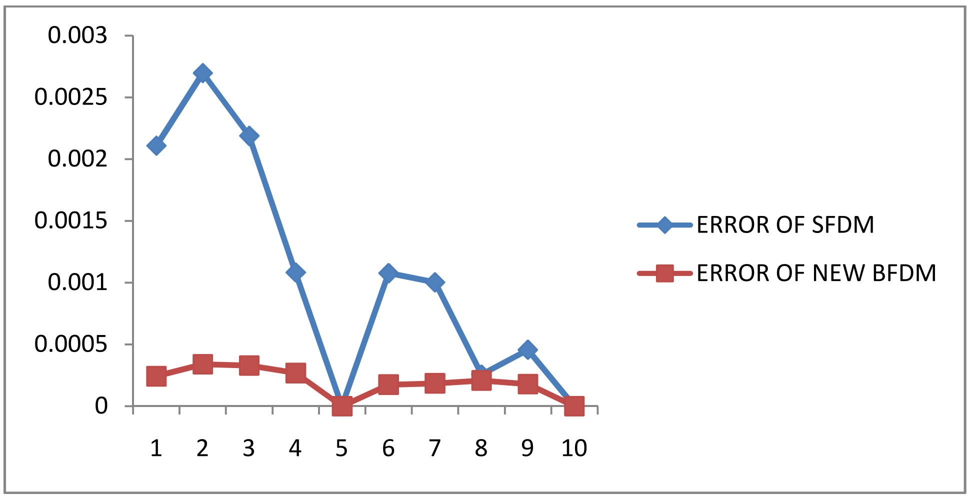

Figure 2: Error graph of Example 2

5.0 Conclusion

The Numerical experiments in this paper shows the results are consistent, convergent to the theoretical solution and compete favorably with the standard finite difference method (See Figures 1 and 2)

References

[1] Soladoye S. O. (2012) Some finite difference methods for the solution of third-order ordinary differential equations (Unpublished) MSc thesis, Nigerian Defence Academy Kaduna.

[2] Onumanyi P. and Ortiz E. I. (1982) Numerical solution of Higher-Order Boundary-value problem for ordinary differential equations with an estimation of the Error. *International Journal of Numerical method for Engineering Volume* ***18,*** *pages 775-781*

[3] Yahaya Y. A. and Onumanyi P. (1997) A symmetric hybrid finite Difference Schemes with Continuous coefficients and its Applications. *Spectrum Journal Volume* ***4,*** *No 1 and 2, pages 198-205*

[4] Yusuph Y. and Onumanyi P. (2002) New multiple FDMS through multi-step Collocation for $y'' = f(x, y)$. *Abacus volume* ***29,*** *pages 92-96*

[5] Yahaya Y. A. and Badmus A. M. (2008) A 3- Step Hybrid Collocation method for the solution of Special third- order initial value problems. *International Journal of Numerical Mathematics Volume* ***3,*** *No 2, pages 306-314.*