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On the Convergence of Differential Equation of Fractional Order

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Abstract

This paper seeks numerical solutions to a fractional differential equation (FDE) with different orders and compares the rate of convergence. It considers Riemann-Liouvielle fractional derivative of different order and uses trapezium rule to obtain an approximate solution. Finally, it implements the method and compares its accuracy and rate of convergence with the Predictor - Corrector method and Extrapolation method.

Keywords: Fractional differential equation, Trapezium rule, Riemann – Liouvielle fractional derivative, Predictor – Corrector method, Extrapolation method.

1.0 Introduction

The solution of fractional differential equations has a major role in the fields of science and engineering. It is one of the fastest growing applications in Mathematical modeling [1].

In the 20th century, fractional derivatives started to find applications in the study of viscoelasticity, diffusion, fluid dynamics, use of composite materials in car manufacture and space exploration problems amongst others. This leads to a need for accurate computational methods to calculate solutions to FDEs. With the advent of computers, numerical methods were developed for calculating the approximate solution to a wider range of FDEs leading to an increase in their application.

In this investigation, a differential equation of fractional order is considered as in Diethelm [2]. Analytical solutions were given as well as numerical solutions. Convergence test were carried out by varying different values for alpha (α) and results were tabulated and the elapsed time were indicated. Matlab programs were used to obtain numerical solutions and the errors obtained are indicated in the table. The Predictor - Corrector method given in Diethelm et al [3] was implemented to get a more accurate solution, the results of which was tabulated and compared to the one in section 3 of this paper. Extrapolation method was employed as expressed in Diethelm [4] to investigate further the rate of convergence of the solutions obtained.

2.0 Governing Equation

Consider the fractional differential equation

$$D^{\alpha}x = -\lambda x, x > 0, \lambda > 0, x(0) = 1,\ 0 \leq \alpha \leq 1 \tag{1.1}$$

First we give the exact solution analytically

When $\alpha = 1$

From (1.1)

$$\frac{dx(t)}{dt} = -\lambda x(t)$$

$$\frac{dx(t)}{x(t)} = -\lambda dt$$

On integration

$$\int_0^t \frac{dx}{x} ds = -\int_0^t \lambda\, ds$$

$$[\ln x(s)]_0^t = -\lambda[s]_0^t$$

$$\ln x(t) - \ln x(0) = -\lambda t$$

$$\ln x(t) - \ln(1) = -\lambda t$$

$$\ln x(t) = -\lambda t$$

$$x(t) = e^{-\lambda t} \tag{1.2}$$

Therefore, the exact solution to (1.1) is $e^{-\lambda t}$

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3.0 Trapezium Rule

We use the trapezium rule to obtain an approximate solution to (1.1) when $\alpha = 1$

$$x_{n+1} = x_n + \frac{h}{2}[f(t_{n+1}, x_{n+1}) + f(t_n, x_n)]$$

$$f(t_n, x_n) = \frac{dx(t)}{dt}$$

$$x_{n+1} = x_n + \frac{h}{2}(-\lambda x_{n+1} - \lambda x_n)$$

$$x_{n+1} = x_n - \frac{\lambda h}{2}x_{n+1} - \frac{\lambda h}{2}x_n$$

$$x_{n+1} + \frac{\lambda h}{2}x_{n+1} = x_n - \frac{\lambda h}{2}x_n$$

$$x_{n+1}\left(1 + \frac{\lambda h}{2}\right) = x_n\left(1 - \frac{\lambda h}{2}\right)$$

$$x_{n+1} = x_n\left[\frac{1 - \frac{\lambda h}{2}}{1 + \frac{\lambda h}{2}}\right]$$

Table 1: Result of numerical and exact solutions when $\alpha = 1$, elapsed time = 0.007677 seconds

T	Numerical Solution	Exact Solution	Error
0.0	1.0	1.0	0.0
0.1	0.9048	0.9048	3.7418e-0.05
0.2	0.8186	0.8187	1.3075e-0.04
0.3	0.7406	0.7408	2.1822e-0.04
0.4	0.6701	0.6703	2.2005e-0.04
0.5	0.6063	0.6065	2.3066e-0.04
0.6	0.5485	0.5488	3.1164e-0.04
0.7	0.4963	0.4966	2.8530e-0.04
0.8	0.4490	0.4493	3.2896e-0.04
0.9	0.4063	0.4066	2.6966e-0.04
1.0	0.3676	0.3679	2.7944e-0.04

4.0 Riemann – Liouvielle Fractional derivative

By the method in Diethelm [2] we obtain a numerical solution of (1.1). Here $D^\alpha x$ denotes the Riemann-Liouvielle fractional derivative of α of the function x.

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_o^t (t-u)^{-\alpha}\, x(u)du \tag{1.3}$$

Based on the algorithm on the observation in [7] that we may interchange differentiation and integration in (1.3) to obtain

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)}\int_o^t (t-u)^{-(\alpha+1)}\, x(u)du \tag{1.4}$$

Where now the integral must be interpreted as a Hadamard finite part integral. Then for a given n, we introduce an equispaced grid $t_j = j/n$ on the interval where the solution of (1.1) is sought. Discretizing with this grid and applying (1.4), we obtain for $j = 1, 2, \ldots\ldots, n$

$$-\lambda x(t) = \frac{1}{\Gamma(-\alpha)}\int_0^{t_j} \frac{x(u) - x(0)}{(t_j - u)^{\alpha+1}}du$$

$$-\lambda x(t) = \frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\int_0^1 \frac{x(t_j - t_j w) - x(0)}{w^{\alpha+1}}dw \tag{1.5}$$

Ignoring the quadrature error, we may solve the resulting equation for the values x_j which will be our approximations for $x(t_j)(j = 1, 2, \ldots\ldots\ldots, n)$. we obtain the following formulas

$$x_j = \frac{1}{w_{oj} - (j/n)^{\alpha}\,\Gamma(-\alpha)\lambda}\left(-\sum_{k=1}^{j} w_{kj}x_{j-k} - \frac{1}{\alpha}x_0\right) \qquad (1.6)$$

With the following weights

$$\alpha(1-\alpha)j^{-\alpha}w_{kj} = \begin{cases} -1 & \text{for } k = 0 \\ 2k^{1-\alpha} - (k-)^{1-\alpha} - (k+1)^{1-\alpha} & \text{for } k = 1, 2, \ldots, j-1 \\ (\alpha-1)k^{-\alpha} - (k-1)^{1-\alpha} + k^{1-\alpha} & \text{for } k = j \end{cases} \qquad (1.7)$$

Table 2: Numerical results when $\alpha = 0.5$, elapsed time = 0.074692 seconds

T	Numerical Solution
0.0	1.0
0.1	0.7811
0.2	0.6809
0.3	0.6186
0.4	0.5741
0.5	0.5396
0.6	0.5117
0.7	0.4883
0.8	0.4683
0.9	0.4509
1.0	0.4355

Clearly, for the two cases, it will take more time to compute the solution in table 2 compared to the solution in table 1 which will take shorter time.

5.0 Predictor - Corrector Method

A more accurate solution can be calculated using the predictor - corrector method given in Diethelm et al [3]. This method is implemented and compared its accuracy and the rate of convergence with the method in section 4.
From [3], the corrector formula is given as

$$x_{k+1} = \sum_{j=0} \frac{t_{k+1}}{j!}x_0^{(j)} + \frac{1}{\Gamma(\alpha)}\left(\sum_{j=0}^{k} a_{j,}k+1 f(t_j, x_j) + a_{k+1,k+1} f(t_{k+1}, x_{k+1}^{p})\right) \qquad (1.8)$$

And the predictor x_{k+1}^{p} is determined by

$$x_{k+1}^{p} = \sum_{j=0}^{[\alpha]-1} \frac{t_{k+1}^{j}}{j!}x_0^{(j)} + \sum_{j=0}^{k} b_{j,k+1} f(t_j, x_j) \qquad (1.9)$$

Where

$$b_{j,k+1} = \frac{h^{\alpha}}{\alpha}\left((k+1-j)^{\alpha} - (k-j)^{\alpha}\right) \qquad (2.0)$$

With the following weights

$$a_{j,k+1} = \frac{h^\alpha}{\alpha(\alpha+1)} \times \begin{cases} (k^{\alpha+1} - (k-\alpha)(k+1)^\alpha) & if\ j = 0 \\ ((k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1}) & if\ 1 \le j \le k \\ 1 & if\ j = k+1 \end{cases} \quad (2.1)$$

These equations are implemented numerically.

Table 3: Numerical results when $\alpha = 0.5$, elapsed time = 0.015684 seconds

T	Numerical Solution
0.0	1.0
0.1	0.5675
0.2	0.4668
0.3	0.4296
0.4	0.4067
0.5	0.3895
0.6	0.3755
0.7	0.3636
0.8	0.3531
0.9	0.3489
1.0	0.3356

Comparing the solutions in table 3 to that of table 2, one finds that the latter is more accurate to the former but the method used in section 5 converges faster than the one in section 4.

6.0 Extrapolation

Extrapolation can be used as part of a numerical scheme to obtain a more accurate approximation by combining previous results. For the system

$$D^\alpha x = -x + t^2 + \frac{2t^{1.5}}{\Gamma(2.5)}, a > 0, \lambda > 0, x(0) = 0 \quad (2.2)$$

For $\alpha = 1$, the Romberg scheme is used to establish how the method works and then for $\alpha = 0.5$, the method in Diethelm and Waltz [4] is used to improve the approximation calculated using the fractional trapezium rule.
The Romberg scheme is given by

$$x_1^{(k)} = x_{i+1}^{k-1} + \frac{x_{i+1}^{(k-1)} - x_i^{(k-1)}}{b^{\lambda\kappa} - 1} \quad (2.3)$$

Now using (1.6), (1.7) and (2.4) below where $b = 2$ and λ_κ is calculated from (2.4), we get the approximation:

$$\lambda_{3j} = 2j + 1 - \alpha, \lambda_{3j-2} = 2j - \alpha. \quad (2.4)$$

Table 4: Numerical results when $\alpha = 1$

Table 5: Numerical results when $\alpha = 1$

T	x0	x1
0.0	1.0	
0.1	0.9048	0.9910
0.2	0.8186	0.8446
0.3	0.7406	0.7641
0.4	0.6701	0.6792
0.5	0.6063	0.6102
0.6	0.5485	0.5520
0.7	0.4963	0.4978
0.8	0.4490	0.4497
0.9	0.4063	0.4069
1.0	0.3676	

Table 5: Numerical results when $\alpha = 0.5$

T	x0	x1
0.0	1.0	1.5285
0.1	0.9048	1.0230
0.2	0.8186	0.8313
0.3	0.7406	0.7260
0.4	0.6701	0.6574
0.5	0.6063	0.6070
0.6	0.5485	0.5682
0.7	0.4963	0.5366
0.8	0.4490	0.5103
0.9	0.4063	0.4881
1.0	0.3676	

7.0 Conclusion

With the results obtained using the different methods discussed in this paper, it has shown that fractional differential equations can be approximated numerically for accuracy and speed. On the results obtained, convergence tests were carried out where discussions and conclusions were drawn for $\alpha = 1$ and $\alpha = 0.5$.

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