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Hybrid Implicit Block Algorithms For Improved Performances In The Solution Of First Order Initial Value Problems

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Abstract

We propose 8- point hybrid implicit linear multistep block method for $k = 4$ with grids and off grid points of $X = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, and\ \frac{7}{2}$ for the solution of first order initial value problems of the form $y' = f(x, y)$. The main integrator in this method is of order 10 and the rest seven integrators are of uniform order 9. All the schemes derived to form our block method come from a single Continuous formulation. The implementation is demonstrated with linear and non linear problems to ascertain their level of accuracy.

Keywords: : (Hybrid, Implicit, methods, Block algorithms, Grid and off grid points, Continuous formulation, first order initial value problems)

1.0 Introduction

Most physical and life problems have been modelled into ordinary differential equations of the form

$$y' = f(x, y),\ y(a) = \propto \qquad (1)$$

There are a number of differential equations which do not possess closed form or finite solutions even if they possess closed form solutions we may not know the analytic way of getting them. Hence there is a great need to develop adequate algorithms of this nature to cater for such problems numerically.

Very many Authors [1],[2],[3],[4],[5] and [6] have contributed immensely in this area.

In this research paper, we develop eight points block methods with four off grid points which are all of uniform order 9 but only the main integrator is of order 10. It implementation show some superiority over existing methods.

Theorem: Well-Posed Condition

Suppose that $f\ and\ f_y$, its first partial derivative with respect to y, are continuous for $x \in [a, b]$. Then the initial value problem $y' = f(x, y),\ a \le x \le b, y(a) = \propto$

has a unique solution $y(x)$ for $a \le x \le b$ and the problem is well posed .(See [6])

Definition 1: Order and Error Constants

A linear multistep method of the form

$$\sum_{j=0}^{k} \propto_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (2)$$

$k \ge 2$ is said to be of order P if $C_0 = C_1 = C_2 = \cdots C_P = 0$

but $C_{P+1} \ne 0$ and C_{P+1} is called error constant

Definition 2: Hybrid method

The hybrid method is the link between the extrapolation method (methods of linear multistep type) and substitution methods (Runge kutta type). We define a K-step hybrid formula as

$$\sum_{j=0}^{k} \alpha_j(x) y_{n+j} = h \sum_{j=0}^{t-1} \beta_j(x) f_{n+j} + h\beta_v(x) f_{n+v} \qquad (3)$$

Definition 3: Explicit and Implicit method

A linear multistep method (1.1.10) is explicit if $\beta_k = 0$ and implicit if $\beta_k \ne 0$

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The main aim of this paper is to generalize the fine work of Branciari [1] and Rhoades [2] by using a pair of weakly compatible maps satisfying a general contractive condition of integral type in a metric space.
We shall need the following definition to prove our results:

Definition 1.3 [6]: A point $p \in X$ is called a coincident point of a pair of self maps S,T if there exists a point q (called a point of coincidence) in X such that $q = Sp = Tp$. Self maps S and T are said to be weakly compatible if they commute at their coincidence points, that is if $Sp = Tp$ for some $p \in X$, then $STp = TSp$.

2.0 Methodology

We consider the contribution of multistep collocation method (MC) with constant step size h, we find a polynomial $y(x)$ of the form

$$y(x) = \sum_{j=0}^{(m+t-1)} \alpha_j(x) y_{n+j} + h \sum_{j=0}^{(m+t-1)} (\beta_j(x) f_{n+j} + \beta_u(x) f_{n+u}) \quad (4)$$

where $u = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ and $\frac{7}{2}$, t and m denotes Interpolation and Collocation points

We consider a continuous solution of the form

$$y(x) - \sum_{j=0}^{m+t-1} a_j x^j \quad (5)$$

$$y'(x) = \sum_{j=1}^{m+t-1} j a_j x^{j-1} \quad (6)$$

where a_j are the parameters to be determined. In this method t=1 and m=9.

We collocate (5) at

$x = x_{n+j}$ $with$ $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2},$ and 4. Also interpolate (4) at $x = x_n$, specifically $k = 4$ yields the following system of non linear equations of the form

$$\sum_{j=0}^{m+t-1} a_j x^j = y_n \quad (7)$$

$$\sum_{j=2}^{m+t-1} j a_j x^{j-1} = f_{n+i} \quad (8)$$

$$i = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4$$

Our proposed continuous formulation takes the form

$$y(x) = \alpha_0(x) y_n + h \Big[\beta_0(x) f_n + \beta_{\frac{1}{2}}(x) f_{n+\frac{1}{2}} + \beta_1(x) f_{n+1} + \beta_{\frac{3}{2}}(x) f_{n+\frac{3}{2}} + \beta_2(x) f_{n+2} + \beta_{\frac{5}{2}}(x) f_{n+\frac{5}{2}} + \beta_3(x) f_{n+3} + \beta_{\frac{7}{2}}(x) f_{n+\frac{7}{2}} + \beta_4(x) f_{n+4} \Big] \quad (9)$$

When using Maple 13 mathematical software to determine the values of α_j and β_j in (8), we obtained the continuous formulation of the form

$$y(x) = y_n + \Big[(x - x_n) - \frac{6849}{2520\,h}(x - x_n)^2 + \frac{29531}{7560\,h^2}(x - x_n)^3 - \frac{2403}{720\,h^3}(x - x_n)^4 + \frac{3207}{1800\,h^4}(x - x_n)^5 - \frac{81}{135h^5}(x - x_n)^6 + \frac{39}{315h^6}(x - x_n)^7 - \frac{9}{630\,h^7}(x - x_n)^8 + \frac{2}{2835h^8}(x - x_n)^9\Big] f_n$$

$$+ \Big[\frac{2520}{315\,h}(x - x_n)^2 - \frac{17316}{945\,h^2}(x - x_n)^3 + \frac{1745}{90\,h^3}(x - x_n)^4 - \frac{2632}{225\,h^4}(x - x_n)^5 + \frac{575}{135h^5}(x - x_n)^6 - \frac{292}{315h^6}(x - x_n)^7 + \frac{35}{315\,h^7}(x - x_n)^8 - \frac{16}{2835h^8}(x - x_n)^9\Big] f_{n+\frac{1}{2}}$$

$$\Big[-\frac{2520}{180h}(x - x_n)^2 + \frac{11178}{270h^2}(x - x_n)^3 - \frac{18353}{360h^3}(x - x_n)^4 + \frac{15289}{450\,h^4}(x - x_n)^5 - + \frac{1790}{135h^5}(x - x_n)^6 + \frac{956}{315h^6}(x - x_n)^7 - \frac{17}{45\,h^7}(x - x_n)^8 + \frac{8}{405h^8}(x - x_n)^9\Big] f_{n+1} +$$

$$\Big[\frac{840}{45h}(x - x_n)^2 - \frac{8012}{135h^2}(x - x_n)^3 + \frac{7173}{90h^3}(x - x_n)^4 - \frac{12864}{225\,h^4}(x - x_n)^5 + \frac{319}{135h^5}(x - x_n)^6 - \frac{1788}{315h^5}(x - x_n)^7 + \frac{33}{45\,h^7}(x - x_n)^8 - \frac{16}{405h^8}(x - x_n)^9\Big] f_{n+\frac{3}{2}} +$$

$$\Big[-\frac{630}{36h}(x - x_n)^2 + \frac{6219}{108h^2}(x - x_n)^3 - \frac{2914}{36h^3}(x - x_n)^4 + \frac{10993}{180\,h^4}(x - x_n)^5 - \frac{7}{27h^5}(x - x_n)^6 + \frac{418}{63h^6}(x - x_n)^7 - \frac{8}{27\,h^7}(x - x_n)^8 + \frac{4}{81h^8}(x - x_n)^9\Big] f_{n+2} +$$

$$\Big[\frac{504}{45h}(x - x_n)^2 - \frac{5076}{135h^2}(x - x_n)^3 + \frac{4891}{90h^3}(x - x_n)^4 - \frac{9544}{225\,h^4}(x - x_n)^5 + \frac{258}{135h^5}(x - x_n)^6 - \frac{1564}{315h^6}(x - x_n)^7 + \frac{31}{45\,h^7}(x - x_n)^8 - \frac{16}{405h^8}(x - x_n)^9\Big] f_{n+\frac{5}{2}} +$$

$$\Big[-\frac{840}{180h}(x - x_n)^2 + \frac{4286}{270h^2}(x - x_n)^3 - \frac{8415}{360h^3}(x - x_n)^4 + \frac{8409}{450\,h^4}(x - x_n)^5 - \frac{1170}{135h^5}(x - x_n)^6 + \frac{732}{315h^6}(x - x_n)^7 - \frac{15}{45\,h^7}(x - x_n)^8 + \frac{8}{405h^8}(x - x_n)^9\Big] f_{n+3} +$$

$$\Big[\frac{8}{7h}(x - x_n)^2 - \frac{412}{105h^2}(x - x_n)^3 + \frac{527}{90h^3}(x - x_n)^4 - \frac{1072}{225\,h^4}(x - x_n)^5 + \frac{61}{27h^5} \;\; \frac{28}{45h^6}(x - x_n)^7 + \frac{29}{315\,h^7}(x - x_n)^8 - \frac{16}{2835h^8}(x - x_n)^9\Big] f_{n+\frac{7}{2}} +$$

$$\Big[-\frac{1}{8h}(x - x_n)^2 + \frac{121}{280h^2}(x - x_n)^3 - \frac{469}{720h^3}(x - x_n)^4 + \frac{967}{1800\,h^4}(x - x_n)^5 - \frac{7}{27h^5}(x - x_n)^6 + \frac{23}{315h^6}(x - x_n)^7 - \frac{1}{90\,h^7}(x - x_n)^8 + \frac{2}{2835h^8}(x - x_n)^9\Big] f_{n+4} \qquad (10)$$

Equation (10) is Our Continuous formulation derived from this methods. Evaluating (10) at $x = x_{n+j}$ $j = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}$ $and\ 4$ to obtain the following eight discrete schemes which form our block method as follows

$$y_{n+4} - y_n = \frac{h}{14175}\Big[1978f_n + 11776f_{n+\frac{1}{2}} - 1856f_{n+1} + 20992f_{n+\frac{3}{2}} - 9080f_{n+2} + 20992f_{n+\frac{5}{2}} - 1856f_{n+3} + 11776f_{n+\frac{7}{2}} + 1978f_{n+4}\Big]$$

$$y_{n+\frac{7}{2}} - y_n = \frac{h}{1036800}\Big[14950f_n + 816634f_{n+\frac{1}{2}} + 48706f_{n+1} + 1085938f_{n+\frac{3}{2}} + 54880f_{n+2} + 736078f_{n+\frac{5}{2}} + 522046f_{n+3} + 223174f_{n+\frac{7}{2}} - 8183f_{n+4}\Big]$$

$$y_{n+3} - y_n = \frac{h}{2800}\Big[401f_n + 2232f_{n+\frac{1}{2}} + 18f_{n+1} + 3224f_{n+\frac{3}{2}} - 360f_{n+2} + 2664f_{n+\frac{5}{2}} + 158f_{n+3} + 72f_{n+\frac{7}{2}} - 9f_{n+4}\Big]$$

$$y_{n+\frac{5}{2}} - y_n = \frac{h}{290304}\Big[41705f_n + 230150f_{n+\frac{1}{2}} + 7550f_{n+1} + 318350f_{n+\frac{3}{2}} - 4000f_{n+2} + 170930f_{n+\frac{5}{2}} - 49150f_{n+3} + 11450f_{n+\frac{7}{2}} - 1225f_{n+4}\Big]$$

$$y_{n+2} - y_n = \frac{h}{28350}\Big[4063f_n + 22576f_{n+\frac{1}{2}} + 244f_{n+1} + 32752f_{n+\frac{3}{2}} - 9080f_{n+2} + 9232f_{n+\frac{5}{2}} - 3956f_{n+3} + 976f_{n+\frac{7}{2}} - 107f_{n+4}\Big]$$

$$y_{n+\frac{3}{2}} - y_n = \frac{h}{89600}\Big[12881f_n + 70902f_{n+\frac{1}{2}} + 3438f_{n+1} + 79934f_{n+\frac{3}{2}} - 56160f_{n+2} + 34434f_{n+\frac{5}{2}} - 14062f_{n+3} + 3402f_{n+\frac{7}{2}} - 369f_{n+4}\Big]$$

$$y_{n+1} - y_n = \frac{h}{226800}\Big[32377f_n + 182584f_{n+\frac{1}{2}} - 42494f_{n+1} + 120088f_{n+\frac{3}{2}} - 116120f_{n+2} + 74728f_{n+\frac{5}{2}} - 31154f_{n+3} + 7624f_{n+\frac{7}{2}} - 833f_{n+4}\Big]$$

$$y_{n+\frac{1}{2}} - y_n = \frac{h}{7257600}\Big[1070017f_n + 4467094f_{n+\frac{1}{2}} - 4604594f_{n+1} + 5595358f_{n+\frac{3}{2}} - 5033120f_{n+2} + 3146338f_{n+\frac{5}{2}} - 1291214f_{n+3} + 312874f_{n+\frac{7}{2}} - 33953f_{n+4}\Big] \quad (11)$$

Equation (11) is our proposed hybrid implicit 8-point block method with the order and Error constants exhibited in table1

Table 1

kSchemes	Order	Error Constants
y_{n+4}	10	$-\frac{37}{14968800}$
$y_{n+\frac{7}{2}}$	9	$\frac{8183}{1061683200}$
y_{n+3}	9	$\frac{9}{1433600}$
$y_{n+\frac{5}{2}}$	9	$\frac{25}{3670016}$
y_{n+2}	9	$\frac{47}{7257600}$
$y_{n+\frac{3}{2}}$	9	$\frac{25}{3670016}$
y_{n+1}	9	$\frac{9}{1433600}$
$y_{n+\frac{1}{2}}$	9	$\frac{8183}{1061683200}$

3.0 Implementation Strategy

The hybrid implicit schemes derived will be implemented in block form with some numerical experiments at $n = 0$, the results produces $y_{\frac{1}{2}}, y_1, y_{\frac{3}{2}}, y_2, y_{\frac{5}{2}}, y_3, y_{\frac{7}{2}}$ and y_4 at once. The advancement of the block method is done at $n = 4, 8, 12, \ldots.., 56$ with hybrid implicit block method. All the results were obtained in block form without overlapping of the solutions and required no starting value.

4.0 Numerical Experiments

The hybrid implicit block method derived is tested with the same linear and non linear problems used in [4] to confirm the

accuracy of the method.

Problem 1

$$y' = -y^2 \quad , y(0) = 1$$

The exact solution is $y(x) = \frac{1}{x+1}$

Table 2: Numerical Results of problem 1 with $h = 0.01$

x	Theoretical solution	Odejide and Adeniran [5] at k=5	New hybrid Implicit Block method at k=4
0.0	1.000000000000000	1.00000000000000	1.000000000000000
0.1	0.909090909090909	0.909090909061729	0.909090909090909
0.2	0.833333333333333	0.833333333296176	0.83333333333334
0.3	0.769230769230769	0.769230769191403	0.769230769230769
0.4	0.714285714285714	0.714285714251721	0.714285714285715
0.5	0.666666666666667	0.666666666637174	0.666666666666668
0.6	0.625000000000000	0624999999973872	0.625000000000000
0.7	0.588235294117647	0.588235294094498	0.588235294117647
0.8	0.555555555555556	0.555562362598403	0.555555555555556
0.9	0.526315789473684	0.526324106923278	0.526315789473685
1.0	0.500000000000000	0.500007506492327	0.500000000000001

Table 3: Comparison of Errors for problem1

Areo et al [1]	Odejide and Adeniran [5]	New hybrid implicit Block method
2.4×10^{-04}	2.91799×10^{-11}	0.00
5.6×10^{-04}	3.71577×10^{-11}	7.0×10^{-15}
7.1×10^{-04}	3.93663×10^{-11}	0.00
8.4×10^{-04}	3.39936×10^{-11}	1.0×10^{-15}
9.6×10^{-04}	2.94922×10^{-11}	1.0×10^{-15}
1.1×10^{-04}	2.61278×10^{-11}	0.00
1.1×10^{-03}	2.31487×10^{-11}	0.00
1.3×10^{-03}	6.80704×10^{-06}	0.00
1.5×10^{-03}	8.31745×10^{-06}	1.0×10^{-15}
1.6×10^{-02}	7.50649×10^{-06}	1.0×10^{-15}

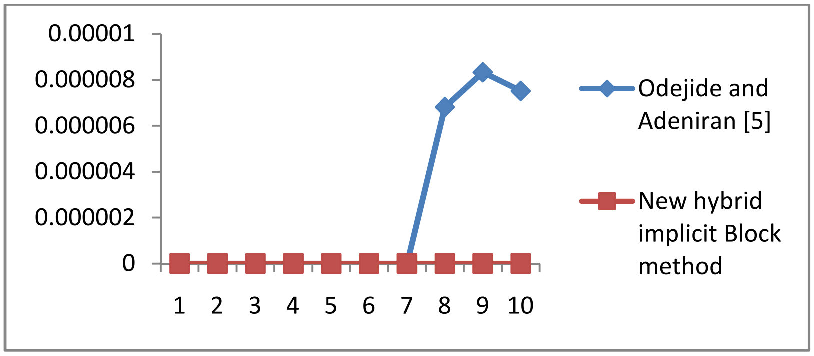

Figure 1: Error graph of solution to problem 1 following data in table 3

Problem 2

$y' = 8(x - y) + 1, \quad y(0) = 2$

The exact solution is $y(x) = x + 2e^{-8x}$

Table 4: Numerical Results of problem 2 with $h = 0.01$

x	Theoretical solution	Odejide and Adeniran [5] at k=5	New hybrid Implicit Block method at k=4
0.0	2.00000000000000	2.00000000000000	2.00000000000000
0.1	0.998657928234443	0.998657928234362	0.998657928234419
0.2	0.603793035989311	0.603793035989697	0.603793035989280
0.3	0.481435906578825	0.481435906578969	0.481435906578805
0.4	0.481524407956732	0.481524407956806	0.481524407956708
0.5	0.536631277777468	0.536631277777507	0.536631277777448
0.6	0.616459494098040	0.616459494098048	0.616459494098010
0.7	0.707395727432966	0.707395727432942	0.707395727432941
0.8	0.803323114546348	0.803323114546321	0.803323114546308
0.9	0.901493171616753	0.901493171616708	0.901493171616721
1.0	1.000670925255805	1.000670925255766	1.000670925255760

Table 5: Comparison of Errors for problem2

Areo et al [1]	Odejide and Adeniran [5]	New hybrid implicit Block method
1.7×10^{-05}	8.07132×10^{-14}	2.4×10^{-14}
1.6×10^{-05}	3.86469×10^{-13}	3.1×10^{-14}
9.3×10^{-06}	1.44384×10^{-13}	2.0×10^{-14}
4.6×10^{-06}	7.30527×10^{-14}	2.4×10^{-14}
1.8×10^{-06}	3.86358×10^{-14}	2.0×10^{-14}
4.2×10^{-07}	7.54952×10^{-15}	3.0×10^{-14}
1.8×10^{-06}	2.34257×10^{-14}	2.5×10^{-14}
2.3×10^{-06}	2.70894×10^{-14}	4.0×10^{-14}
3.8×10^{-07}	4.57412×10^{-14}	3.2×10^{-14}
3.2×10^{-07}	3.95239×10^{-14}	4.0×10^{-14}

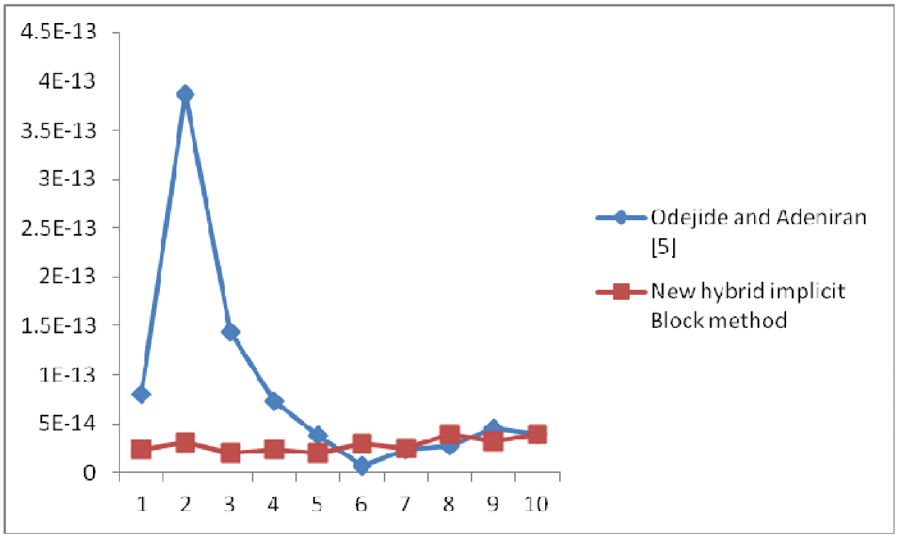

Figure 2: Error graph of solution to problem 2 following data in table 5

5.0 Conclusion

We want to conclude that the new hybrid implicit block method is consistent, stable and converges to the exact solution with the two problems tested. Also the results obtained from the two problems shows its superiority over Odejide and Adeniran [5]. See error graph of figures 1 and 2

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