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On The Region of Absolute Stability of A New Numerical Scheme For The Solution Of Initial Value Problems In Ordinary Differential Equations

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Abstract

This paper presents the detail analysis of the region of absolute stability (RAS) of a new numerical scheme suitable for solving oscillatory system. The work also presents the Comparison of the RAS analyzed with other existing ones and a relationship was established which we present as a theorem.

1.0 Introduction

Over the years, many Numerical integrating schemes to generate the numerical solutions to problems in Ordinary Differential Equations (ODE) have been developed by several authors [1 – 7]. Generally the efficiency of any of the methods depends on the method's stability and certain accuracy properties. The accuracy properties of different methods are usually compared by considering the order of convergence as well as the truncation error coefficient of the various methods.

[7] and [8] is an improvement on [3], [4] and [2]. [8] Proposed a numerical integration scheme of order six which is particularly well suited to solve initial value problem having oscillatory or exponential solutions. This method was based on the local representation of the theoretical solution y(x) to the initial value problem of the form

$$y' = f(x, y), \; y(a) = h \tag{1.1}$$

In the interval (x_t, x_{t+1}) by a polynomial interpolating function

$$F(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + b\,\mathrm{Re}(e^{rx+m}) \tag{1.2}$$

Where a_0, a_1, a_2, a_3 *and* b are real undetermined coefficients, while r and m are complex parameters.

1.2 The Basic Interpolant

Let us assume that the theoretical solution y(x) to the initial value problem (1.1) can be locally represented in the interval $(x_t, x_{t+1}), t^3 \;\; 0$ by the polynomial interpolating function (1.2). If we put,

$$r = r_1 + ir_2 \tag{1.5}$$

and $m = is$, $i^2 = -1$ in (1.2), we obtain the following interpolating function;

$$F(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + be^{r_1 x}\cos(r_2 x + s) \tag{1.6}$$

Let

$$R(x) = e^{r_1 x} \tag{1.7}$$

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and

$$q(x) = r_2 x + s \tag{1.8}$$

Then, we obtain

$$F(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + bR(x)\cos q(x) \tag{1.9}$$

We shall assume that y_t is a numerical estimate to the theoretical solution $y(x_t)$ and that $f_t = f(x_t, y_t)$. We define mesh points as follows

$$x_t = a + th \tag{1.10}$$

t=0, 1, 2, 3…

With some imposed constraints, [8] came out with a scheme of order six,

$$
\begin{aligned}
y_{t+1} = y_t &+ \Bigg\{ f_t - \Bigg[f_t^{(1)} - \Bigg(f_t^{(2)} - \frac{[(r_1^3 - 3r_1 r_2^2)\cos q_t + (r_2^3 - 3r_1^2 r_2)\sin q_t] f_t^{(3)}}{[(r_1^4 + r_2^4 - 6r_1^2 r_2^2)\cos q_t + (4r_1 r_2^3 - 4r_1^3 r_2)\sin q_t]} \Bigg) \Bigg](a+th) \\
&- \frac{1}{2}\Bigg[f_t^{(2)} - \frac{[(r_1^3 - 3r_1 r_2^2)\cos q_t + (r_2^3 - 3r_1^2 r_2 \sin q_t)] f_t^{(3)}}{[(r_1^4 + r_2^4 - 6r_1^2 r_2^2)\cos q_t + (4r_1 r_2^3 - 4r_1^3 r_2)\sin q_t]} \Bigg](a+th) \\
&- \Bigg[\frac{[r_1 \cos q_t - r_2 \sin q_t] f_t^{(3)}}{[(r_1^4 + r_2^4 - 6r_1^2 r_2^2)\cos q_t + (4r_1 r_2^3 - 4r_1^3 r_2)\sin q_t]} \Bigg] \Bigg\} h \\
&+ \frac{1}{2}\Bigg\{ f_t^{(1)} - \Bigg[f_t^{(2)} - \frac{[(r_1^3 - 3r_1 r_2^2)\cos q_t + (r_2^3 - 3r_1^2 r_2)\sin q_t] f_t^{(3)}}{[(r_1^4 + r_2^4 - 6r_1^2 r_2^2)\cos q_t + (4r_1 r_2^3 - 4r_1^3 r_2)\sin q_t]} \Bigg](a+th) \\
&- \Bigg[\frac{[(r_1^2 - r_2^2)\cos q_t - 2r_1 r_2 \sin q_t] f_t^{(3)}}{[(r_1^4 + r_2^4 - 6r_1^2 r_2^2)\cos q_t + (4r_1 r_2^3 - 4r_1^3 r_2)\sin q_t]} \Bigg] \Bigg\} (2ah + (1+2t)h^2) \\
&+ \frac{1}{6}\Bigg\{ \Bigg[f_t^{(2)} - \frac{[(r_1^3 - 3r_1 r_2^2)\cos q_t + (r_2^2 - 3r_1^2 r_2)\sin q_t] f_t^{(3)}}{[(r_1^4 + r_2^4 - 6r_1^2 r_2^2)\cos q_t + (4r_1 r_2^3 - 4r_1^3 r_2)\sin q_t]} \Bigg] \\
&\quad \big(3a^2 h + ah^2(3+6t) + h^3(3t^2 + 3t + 1)\big) \\
&+ \Bigg[\frac{[e^{r_1 h}(\cos q_t \cos r_2 h - \sin q_t \sin r_2 h) - \cos q_t] f_t^{(3)}}{[(r_1^4 + r_2^4 - 6r_1^2 r_2^2)\cos q_t + (4r_1 r_2^3 - 4r_1^3 r_2)\sin q_t]} \Bigg] \Bigg\}
\end{aligned} \tag{1.11}
$$

DERIVATION OF THE STABILITY FUNCTION

To obtain the stability function for the new scheme (1.11), we proceed as follows;

Put

$$
\begin{aligned}
A_1 = \Bigg\{ f_t &\Bigg[f_t^{(1)} - \Bigg(f_t^2 - \frac{[(r_1^3 - 3r_1 r_2^2)\cos q_t + (r_2^3 - 3r_1^2 r_2)\sin q_t] f_t^{(3)}}{[(r_1^4 + r_2^4 - 6r_1^2 r_2^2)\cos q_t + (4r_1 r_2^3 - 4r_1^3 r_2)\sin q_t]} \Bigg) \Bigg](a+th) \\
&- \frac{1}{2}\Bigg[f_t^{(2)} - \frac{[(r_1^3 - 3r_1 r_2^2)\cos q_t + (r_2^3 - 3r_1^2 r_2)\sin q_t] f_t^{(3)} (a+th)^2}{[(r_1^4 + r_2^4 - 6r_1^2 r_2^2)\cos q_t + (4r_1 r_2^3 - 4r_1^3 r_2)\sin q_t]} \\
&+ \Bigg[\frac{(r_1 \cos q_t - r_2 \sin q_t) f_t^{(3)}}{[(r_1^4 + r_2^4 - 6r_1^2 r_2^2)\cos q_t + (4r_1 r_2^3 - 4r_1^3 r_2)\sin q_t]} \Bigg] \Bigg\} h
\end{aligned}
$$

$$A_2 = \frac{1}{2}\left\{ f^{(1)}{}_t - \left[f_t^{(2)} - \frac{[(r_1^3 - 3r_1 r_2^2)\cos\theta_t + (r_2^3 - 3r_1^2 r_2)\sin\theta_t] f_t^{(3)}(a+th)}{[(r_1^4 + r_2^4 - 6r_1^2 r_2^2)\cos\theta_t + (4r_1 r_2^3 - 4r_1^3 r_2)\sin\theta_t]} \right] \right.$$
$$\left. - \left[\frac{[(r_1^2 - r_2^2)\cos\theta_t - 2r_1 r_2 \sin\theta_t] f_t^{(3)}}{[(r_1^4 + r_2^4 - 6r_1^2 r_2^2)\cos\theta_t + (4r_1 r_2^3 - 4r_1^3 r_2)\sin\theta_t]} \right] \right\} (2ah + (1+2t)h^2$$

$$A_3 = \frac{1}{6}\left\{ \left[f_t^{(2)} - \frac{[(r_1^3 - 3r_1 r_2^2)\cos\theta_t + (r_2^3 - 3r_1^2 r_2)\sin\theta_t] f_t^{(3)}}{[(r_1^4 + r_2^4 - 6r_1^2 r_2^2)\cos\theta_t + (4r_1 r_2^3 - 4r_1^3 r_2)\sin\theta_t]} \right] \right.$$
$$\left. (3a^2 h + ah^2(3+6t) + h^3(3t^2 + 3t + 1) \right\}$$

$$A_4 = \left[\frac{(e^{r_1 h}(\cos\theta_t \cos r_2 h - \sin\theta_t \sin r_2 h) - \cos\theta_t) f_t^{(3)}}{(r_1^4 + r_2^4 - 6r_1^2 r_2^2)\cos\theta_t + (4r_1 r_2^3 - 4r_1^3 r_2)\sin\theta_t} \right]$$

We proceed to expand the above terms with the fact that

$$f(x,y) = \lambda y$$
$$f_t = \lambda y_t$$
$$f_t^1 = \lambda f_t = \lambda(\lambda y_t) = \lambda^2 y_t$$
$$f_t^2 = \lambda f_t^1 = \lambda^3 y_t$$
$$f_t^3 = \lambda^4 y_t$$

and the expression of $e^{r_1 h}, \cos r_2 h$ and $\sin r_2 h$ are defined as ,

$$\cos(r_2 h) = \sum_{r=0}^{\infty} (-1)^r \frac{(r_2 h)^{2r}}{(2r)!}$$
$$\sin(r_2 h) = \sum_{r=0}^{\infty} (-1)^r \frac{(r_2 h)^{2r+1}}{(2r+1)!}$$

and

$$e^{r_1 h} = \sum_{r=0}^{\infty} \frac{(r_1 h)^r}{r!}$$

then

On simplification, equation (1.1) yields

$$y_{t+1} = y_t + \lambda h y_t + \frac{1}{2}h^2\lambda^2 y_t + \frac{1}{6}h^3\lambda^3 y_t$$
$$\Rightarrow \quad y_{t+1} = y_t(1 + \lambda h + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3)$$
$$\frac{y_{t+1}}{y_t} = (1 + \lambda h + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3)$$

We define, $z = \lambda h$

We have

$$m(z) = \frac{y_{t+1}}{y_t} = (1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3)$$

Therefore

$$m(z) = \left[1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3\right]$$

Is the stability polynomial

Definition 2.1

The One step scheme is said to be absolutely stable at a point Z in the complex plane provided the stability function or polynomial $m(z)$ fulfils the condition

$|m(z)| < 1$

Definition 2.2

A region D of the complex plane is said to be a REGION OF ABSOLUTE STABILITY (RAS) of a system if the method is absolutely stable for every z$\in$D consequently D will be regarded as a region of absolute stability for a one-step integrator if $|m(z)| < 1$

For z$\in$D. That is

$RAS = \{z : | m(z) | < | \}$

With the Stability Polynomials of [7] scheme

$$\mu(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!}$$

$$= 1 + (u + iv) + \frac{(u + iv)^2}{2} + \frac{(u + iv)^3}{6}, \quad where \quad z = u + iv$$

$$= 1 + u + iv + \frac{u^2 + i2uv - v^2}{2} + \frac{u^3 + i3u^2v - 3uv^2 - v^3}{6}$$

$$= \frac{6 + 6u + i6v + 3u^2 + i6uv - 3v^2 + u^3 + i3u^2v - 3uv^2 - iv^3}{6}$$

$$= \frac{6 + 6u + 3u^2 - 3v^2 + u^3 - 3uv^2}{6} + i\,\frac{6v + 6uv + 3u^2v - v^3}{6}$$

$$= A(u, v) + iB(uv)$$

$$|\mu(u, v)| \le 1 \Leftrightarrow A^2(u, v) + B^2(u, v) \le 1$$

Which in turn holds if and only if

$$(6 + 6u + 3u^2 - 3v^2 + u^3 - 3uv^2)^2 + (6v + 6uv + 3u^2v - v^3)^2 \le 36$$

$$36 + 72u + 72u^2 + 48u^3 + (21u^2 - 3v^2)(u^2 + v^2) + 6u(u^2 + v^2)^2 + (u^2 + v^2)^3 \le 36$$

$$employing \;\; u = R\cos\theta, v = R\sin\theta$$

$$72R\cos\theta + 72R^2\cos^2\theta + 48R^3\cos^3\theta + (21\cos^2\theta - 3\sin^2\theta)R^4 + 6\cos R^5 + R^6 \le 0$$

$\therefore$ *our Jordan Curve* $F(R,\theta)=0$ *becomes*

$$F(R,\theta) = R\begin{bmatrix} 72\cos\theta + 72R\cos^2\theta + 48R^2\cos^3\theta + (21\cos^2\theta - 3\sin^2\theta)R^3 \\ \\ +6(\cos\theta R^4 + R^5 \end{bmatrix}$$

$At \quad \theta = 0^0,$

$$F(R,0^0) = R\left[72 + 72R + 48R^2 + 21R^3 + 6R^4 + R^5\right]$$

$$= 0 \Leftrightarrow R = 0$$

$$OR \qquad R^5 + 6R^4 + 21R^3 + 48R^2 + 72R + 72 = 0$$

$$\begin{aligned} F(-2,0) \quad &= \quad (-2)\left[72 - 144 + 48(-2)^2 + 21(-2)^3 + 6(-2)^4 + (-2)^5\right] \\ &= \quad (-2)\left[72 - 144 + 192 - 168 + 96 - 32\right] \\ &= \quad (-2)\left[(72 + 192 + 96) - (144 + 168 + 32)\right] \\ &= \quad (-2)\left[360 - 344\right] \\ &= \quad (-2)(16) < 0 \end{aligned}$$

$so \quad (-2,0) \quad is \quad an \quad \text{int}\,erior \quad po\,\text{int}$

$$\begin{aligned} F(-3,0) \quad &= \quad (-3)\left[72 - 216 + 48(-3)^2 + 21(-3)^3 + 6(-3)^4 + (-3)^5\right] \\ &= \quad (-3)\left[72 - 216 + 432 - 567 + 486 - 243\right] \\ &= \quad (-3)\left[(72 + 432 + 486) - (216 + 567 + 243)\right] \\ &= \quad (-3)\left[990 - 1026\right] \\ &= \quad (-3)(-36) > 0 \end{aligned}$$

$\therefore (-3,0) \quad is \quad an \quad exterior \quad po\,\text{int}\,of \quad F(R,\theta)$

$\therefore \exists$

a point $\tau \in (-3,-2)$ such that

$$F(\tau,0) = 0$$

Which implies that τ is closer to -2 than it is to -3
Hence IAS = (τ , 0) where $\tau \in$ (-3,-2) confirming the conformity of [7] with the graphical solution.

$$R\left[72(\frac{\sqrt{3}}{2}) + 72R(\frac{3}{4}) + 48R^2(\frac{\sqrt{3}}{2})^3 + (21(\frac{3}{4}) - 3(\frac{1}{4}))R^3 + 6(\frac{\sqrt{3}}{2})R^4 + R^5\right]$$

$$= R\left[36\sqrt{3} + 54R + 18\sqrt{3}R^2 + 15R^3 + 3\sqrt{3}R^4 + R^5\right] = F(R,30^0)$$

$$F(-3,30^0)=(-3)\left[441\sqrt{3}-810\right]>0$$

$$\therefore(-3,30^0)\ \ is\ \ an\ \ exterior\ \ point\ \ of\ \ F(R,\theta)=0$$

$$Next\ \ is\ \ F(-2,30^0)=(-2)\left[156\sqrt{3}-260\right]<0$$

$$\Rightarrow(-2,30^0) is\ \ an\ \ interior\ \ point\ \ of\ \ F(R,\theta)=0$$

$$And\ \ so\ \ \exists\tau\in(-3,2)\ni\ \ F(\tau,30^0)=0$$

$$just\ \ like\ \ Ibijola[4]\ \ when\ \ \theta=45^0\ \ we\ \ have$$

$$F(R,45^0)=R\left[36\sqrt{2}+36R+12\sqrt{2}R^2+9R^3+3\sqrt{2}R^4+R^5\right]$$

$$this\ \ leads\ \ us\ \ to$$

$$F(-3,45^0)>0\ \ and\ \ F(-2,45^0)>0,\ \ F(\tau,45^0)=0$$

$$where\ \ (\tau,45^0)\ \ is\ \ on\ \ the\ \ Jordan\ \ curve\ \ F(R,\theta)=0$$

$$At\ \ \theta=60^0\ \ we\ \ obtain\ \ (note\ \ \sin 60^0=\frac{\sqrt{3}}{2},\ \ \cos 60^0=\frac{1}{2})$$

$$F(R,60^0)=R\left[36+18R+6R^2+3R^3+3R^4+R^5\right]$$

$$hence$$

$$F(-3,60^0)>0,\ \ F(-2,60^0)<0\ \ \Rightarrow\exists$$

$$\ni\ \ F(\tau,60^0)=0$$

$$At\ \ \theta=90^0, we\ \ have$$

$$F(R,90^0)=R^4\left[R^2-3\right]$$

$$Next\ \ is\ \ \theta=135^0, Note\ \ \cos 135=-\cos 45=-\frac{\sqrt{2}}{2}, \sin 135=\sin 45=\frac{\sqrt{2}}{2}$$

$$\Rightarrow F(R,135^0)=R\left[72\cos(-\frac{\sqrt{2}}{2})+72R(-\frac{\sqrt{2}}{2})^2+48R^2(-\frac{\sqrt{2}}{2})^3+\{21(-\frac{\sqrt{2}}{2})^2-3(\frac{\sqrt{2}}{2})^2\}R^3+6(-\frac{\sqrt{2}}{2})R^4+R^5\right]$$

$$=R\left[-36\sqrt{2}+36R-12\sqrt{2}R^2+\{\frac{21-3}{2}\}R^3-3\sqrt{2}R^4+R^5\right]$$

$$\therefore F(R,135^0)=R\left[-36\sqrt{2}+36R-12\sqrt{2}R^2+9R^3-3\sqrt{2}R^4+R^5\right]$$

$$\Rightarrow F(3,135^0)=3\left[-36\sqrt{2}+36(3)-12\sqrt{2}(3)^2+9(3)^3-3\sqrt{2}(3)^4+(3)^5\right]>0$$

$$and \quad so(3,135^0) \quad is \quad an \quad exterior \quad po\text{int} \quad of \quad F(R,\theta)=0$$

$$similarly \quad F(2,135)>0 \Rightarrow (2,135) \quad is \quad an \quad \text{int}erior \quad po\text{int} \quad of \quad F(R,\theta)=0$$

$$\therefore \exists \quad a \quad po\text{int} \quad \tau \in (2,3) \quad \ni F(\tau,135^0)=0$$

$$Next \quad is \quad F(R,150^0), \quad which, \quad by \cos 150 = -\cos 30 \quad and \quad \sin 150^0 = \sin 30^0)$$

$$F(R,150^0) = R\left[72(\frac{\sqrt{3}}{2}) + 72R(\frac{3}{2})^2 + 48R^2(\frac{\sqrt{3}}{2})^3 + (21(\frac{3}{2})^2 - 3(\frac{1}{2})^2)R^3 + 6(\frac{\sqrt{3}}{2})R^4 + R^5\right]$$

$$= R\left[-36\sqrt{3} + 54R - 18\sqrt{3}R^2 + \{\frac{63-3}{4}\}R^2 - 3\sqrt{3}R^4 + R^5\right]$$

$$\therefore F(R,135^0) = R\left[-36\sqrt{3} + 54R - 18\sqrt{3}R^2 + 15R^2 - 3\sqrt{3}R^4 + R^5\right]$$

From where we get

$$F(3,150) = 3\left[-441\sqrt{3}+810\right] > 0 \quad and \quad F(2,150^0) = 2\left[-156\sqrt{3}+260\right] < 0$$

$$i.e \quad (3,150^0) \quad is \quad an \quad enterior \quad po\text{int} \quad of \quad F(R,\theta)=0 \quad while$$

$$(2,150^0) \quad is \quad an \quad \text{int}erior \quad po\text{int} \quad of \quad F(R,\theta)=0$$

$$hence \quad \exists \tau \in (2,3) \ni F(\tau,150^0)=0$$

$$Next \quad is \quad 180^0 \quad which \quad gives \quad \cos 180^0 = -1 \quad and \quad \sin 180^0 = 0$$

$$i.e \quad F(R,180^0) = R\left[-72 + 72R - 48R^2 + 21R^3 - 6R^4 + R^5\right]$$

$$here, \quad as \quad in \quad the \quad previous \quad cases, \quad we \quad obtain$$

$$F(3,180^0) = 3(36) > 0, \quad F(2,180^0) = 2(-16) < 0$$

$$\Rightarrow (3,180^0),(2,180^0) \quad are \quad anterior, \text{int}erior \quad po\text{int}s$$

$$of \quad F(R,\theta) \quad respectively.$$

$$consequently \quad \exists \tau \in (2,3) \ni \quad F(\tau,180^0)=0$$

We find from elementary trigonometric ratios that;

$$F(R,210^0) = F(R,150^0)$$
$$F(R,225^0) = F(R,135^0)$$
$$F(R,240^0) = F(R,120^0)$$
$$F(R,270^0) = F(R,90^0)$$
$$F(R,300^0) = F(R,60^0)$$
$$F(R,315^0) = F(R,45^0)$$
$$F(R,330^0) = F(R,30^0)$$
$$F(R,360^0) = F(R,0^0)$$

We have at each point of the angle $\theta \in [0,360)$

$$\left|R[3]\right| \le \left|R[2]\right| \le \left|R[7]\right|$$

Hence
RAS [3] $\subseteq$ RAS [2] $\subseteq$ RAS [7]

THEOREM
The RAS of [2] is a sub region of the RAS of [7].

Proof
The stability function of [2] is

$$\mu(z) = 1 + z + z^2$$

$$= \frac{1}{2}(2 + 2u + u^2 - v^2) + i(uv + v), i = \sqrt{-1}$$

By employing the transformation
$u = R\cos\theta$ and $v = R\sin\theta$
We obtain

$$\left|\mu(R,\theta)\right| \le 1$$

$$R\left[R^3 + 4R^2\cos\theta + 8R\cos^2\theta + 8\cos\theta\right] \le 0$$

Define

$$F(R,\theta) = R\left[R^3 + 4R^2\cos\theta + 8R\cos^2\theta + 8\cos\theta\right]$$

Then,
$F(R,\theta) = 0$ Is the Jordan curve of the RAS. In this case,
$F(R,\theta) < 0$ representing the interior of the Jordan curve is the RAS of the [2] while $F(R,\theta) > 0$ representing the exterior of the Jordan curve is the RIS of [2].
For [7], the stability function

$$\mu(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6}$$

yielding

$$\mu(u,v) = \frac{6 + 6u + 3u^2 - 3v^2 + u^3 - 3uv^2}{6} + i\frac{6v + 6uv + 3u^2v\_v^3}{6}, i = \sqrt{-1}$$

The use of the polar transformation

$$u = R\cos\theta, \quad v = R\sin\theta$$

$$leads \quad us \quad to$$

$$\left|\mu(R,\theta)\right| \le 1$$

$$if \quad and \quad only \quad if$$

$$R\left[\begin{array}{l} R^5 + 6R^4\cos\theta + \{21\cos^2\theta - 3\sin^2\theta\}R^3 + 48R^2\cos^3\theta + \\ 72R\cos^2 + 72\cos\theta \end{array}\right] \le 0$$

Our Jordan curve $F(R,\theta) = 0$ in this case becomes

$$F(R,\theta) = R\left[\begin{array}{l} R^5 + 6R^4\cos\theta + \{21\cos^2\theta - 3\sin^2\theta\}R^3 + 48R^2\cos^3\theta + 72R\cos^2 + \\ 72\cos\theta \end{array}\right]$$

Hence $F(R,\theta) < 0$ representing the interior of the Jordan curve is the RAS of [7] while $F(R,\theta) > 0$ representing the exterior of the Jordan curve is the RIS of [7].

The curves of the two functions are both located in the second and third quadrants of the U-V plane. They both pass through the origin.

On [2] the curve has its radial value zero at 90^0 and at 270^0. From these two points, the radial value rise in absolute variationally to its peak at 180^0. If we denote the respective radial values by
R [2] to mean the radial value of the Jordan curve of [2] at each angle $\theta \in [0,360]$
R [7] to mean the radial value of the Jordan curve of [7] at each point
$\theta \in [0,360]$
Then we have

$$\left|R[2]\right| \le \left|R[7]\right|$$

$$Hence$$

$$\left\{(R,\theta) : R\left[R^3 + 4R\cos\theta + 8R\cos^2\theta + 8\cos\theta\right] \le 0, \quad 0 \le \theta \le 360^0\right\}$$

$$\frac{c}{\neq}\left\{(R,\theta) : R\left[\begin{array}{l} R^5 + 6R^4\cos\theta + \{21\cos^2\theta - 3\sin^2\theta\}R^3 + 4R^2\cos^3\theta \\ +72R\cos^2\theta + 72\cos\theta \le 0 \end{array}\right], 0 \le \theta \le 360^0\right\}$$

$$i.e \quad RAS[2] \subset RAS[7]$$

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