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The Finite Deformation of Internally pressurized Hollow Cylindrical Pipe of a Blatz-ko Material

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Abstract

Finite deformation of a hollow cylindrical pipe of a Blatz-Ko material deforming under applied uniform internal pressure is studied. The analysis resulted into a non-linear second order ordinary differential equation for the determination of displacements. An asymptotic solution of this is sought in the Sobolev Space $W^{1,2}$ and approximate closed form solutions of stresses and displacements are obtained. The explicit form of the stress solution makes further investigation of the concept at various pressure levels easy.

Keywords: Compressible; Sobolev Space; Asymptotic Solution; Norm.

1.0 Introduction

Exact solutions to problems of determining stresses and displacements in a homogeneous and isotropic linearly elastic hollow cylinder under internal pressure are well known. Abeyaratne and Horgan [1] obtained a parametric solution to a problem describing finite plane strain deformation of an infinite medium of Blatz-ko material (a nonlinear elastic material). The medium considered was an infinite medium with a circular cylindrical cavity under pressure loading condition. Later, Chung et al [2] applied the same technique to solve problems of pressurized hollow cylinders and spheres with finite radii.

Recently, Ejike and Erumaka [3] obtained an exact solution to the problem of a rotating circular cylinder of a nonlinear elastic incompressible material. They used an asymptotic approach and sought for the solution in the Sobolev space $W^{1,2}$. Note that the problem involves the minimization of approximate errors and this is best executed in Sobolev space where it is possible to minimize not only the solution functions but also their gradients. Erumaka. [4 – 5] respectively applied the same method and obtained approximate solutions to the problem of rotating hollow and solid spheres rotating about their axes with constant angular velocity.

In this paper we apply the asymptotic method in the Sobolev space $W^{1,2}$ to provide an approximate solution to the problem of a hollow cylinder of Blatz-ko material under a uniformly distributed internal pressure with a traction free surface.

2.0 Formulation of Boundary Value Problem

Let

$$\Omega_0 = \{(r, \theta, z) : a \le r \le b,\ 0 \le \theta \le 2\pi,\ 0 \le z \le h\} \qquad (2.1)$$

denote the cross-section of the right circular hollow cylinder in the undeformed configuration. The deformation which takes the point (r, θ, z) of the undeformed configuration to the point (R, θ, Z) of the deformed configuration $\Omega = \{(R, \theta, Z)\}$ is given by

$$\begin{aligned} &R = R(r) && a \le r \le b \\ &\theta = \theta && 0 \le \theta \le 2\pi \\ &Z = z && 0 \le Z \le h, \end{aligned} \qquad (2.2)$$

where $R(r) \in C^2(a, b)$ and is such that R and its derivative $\dot{R}(r)$ are non negative. We have assumed that the cylinder is large and long enough that end effect is negligible.

The deformation gradient tensor $\overline{F}$ is given by

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$$\overline{F} = \begin{pmatrix} \dot{R} & 0 & 0 \\ 0 & \frac{R}{r} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.3}$$

The left Cauchy – Green deformation gradient tensor $\overline{B}$ is given by

$$\overline{B} = \overline{F}\ \overline{F}^T = \begin{pmatrix} \dot{R}^2 & 0 & 0 \\ 0 & \left(\frac{R}{r}\right)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.4}$$

where $\overline{F}^T$ denotes the transpose of $\overline{F}$. The strain invariants are

$$I_1 = \text{tr } B = \dot{R}^2 + \left(\frac{R}{r}\right)^2 + 1 \tag{2.5a}$$

$$I_2 = \tfrac{1}{2}\left((tr\overline{B})^2 - tr\overline{B}^2\right) = \left(\frac{R\dot{R}}{r}\right)^2 + \dot{R}^2 + \left(\frac{R}{r}\right)^2 \tag{2.5b}$$

and

$$I_3 = \det \overline{B} = \left(\dot{R}\frac{R}{r}\right)^2, \tag{2.5c}$$

where tr $\overline{B}$ = trace of $\overline{B}$ and det $\overline{B}$ = determinant of $\overline{B}$
Now let
$W = W(I_1, I_2, I_3)$
be the strain energy density function of the material considered, then the stress field for the compressible material is given in Atkin and Fox [6] as

$$\overline{\tau} = \psi_0 I + \psi_1 \overline{B} + \psi_{-1} \overline{B}^{-1}, \tag{2.6}$$

where the coefficients are given in terms of W as

$$\psi_0 = 2I_3^{-1/2}(I_2W_2 + I_3W_3); \tag{2.7a}$$
$$\psi_1 = 2I_3^{1/2} W_1; \tag{2.7b}$$
$$\psi_{-1} = -2I_3^{-1/2} W_2; \tag{2.7c}$$

I is the unit tensor and

$$W_i = \frac{\partial W}{\partial I_i} \quad i = 1, 2, 3 \tag{2.8}$$

Here, we consider a particular homogeneous isotropic compressible elastic material, namely the Blatz-ko material. The Blatz-ko material is characterized by the strain energy density function

$$W = \frac{\mu}{2}(I_2I_3^{-1} + 2I_3^{1/2} - 5), \tag{2.9}$$

where μ is the shear modulus of the material at infinitesimal deformation.
It is easy to see that for the material in equation (2.9) the stress tensor (2.6) reduces to

$$\overline{\tau} = \psi_0 I + \psi_{-1} \overline{B}^{-1} \tag{2.10}$$

Now using equations (2.7) – (2.9), we obtain the non-trivial components of the non-trivial stresses as the principal stress components

$$\tau_{RR} = \mu\left(1 - \frac{r}{R\dot{R}^3}\right), \tag{2.11a}$$

$$\tau_{\theta\theta} = \mu\left(1 - \frac{r^3}{\dot{R}R^3}\right), \tag{2.11b}$$

$\tau_{zz} = \mu\left(1 - \frac{r}{R\dot{R}}\right)$, (2.11c)

$\tau_{R\theta} = \tau_{RZ} = \tau_{\theta Z} = \tau_{Z\theta} = \tau_{\theta Z} = 0$ $a < r < b$ (2.11d)

In the absence of body force, the equilibrium equation is given by

$$\text{div } \bar{\tau} = 0 \tag{2.12}$$

Equation (2.12) in components form is equivalent to the three equations

$$\frac{\partial\tau_{RR}}{\partial R} + \frac{1\partial\tau_{R\theta}}{R\partial\theta} + \frac{\partial\tau_{Rz}}{\partial z} + \frac{1}{R}(\tau_{RR} + \tau_{\theta\theta}) = 0$$

$$\frac{\partial\tau_{\theta R}}{\partial R} + \frac{1}{R}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{\partial\tau_{\theta z}}{\partial z} + \frac{2}{R}\tau_{\theta R} = 0$$

$$\frac{\partial\tau_{zR}}{\partial R} + \frac{1}{R}\frac{\partial\tau_{z\theta}}{\partial\theta} + \frac{\partial\tau_{zz}}{\partial z} + \frac{2}{R}\tau_{zr} = 0$$

In the present case (2.12) reduces to a single equation

$$\frac{d\tau_{RR}}{dR} + \frac{1}{R}(\tau_{RR} + \tau_{\theta\theta}) = 0$$

Now

$$\frac{d\tau_{RR}}{dR} = \frac{d\tau_{RR}}{dR}\frac{dr}{dR} = \frac{1}{\dot{R}}\frac{d\tau_{RR}}{dr}$$

So that the equilibrium equation becomes

$$\frac{d\tau_{RR}}{dr} + \frac{\dot{R}}{R}(\tau_{RR} + \tau_{\theta\theta}) = 0 \quad a < r < b \tag{2.13}$$

Equation (2.13) together with (2.11) yields the following nonlinear second-order ordinary differential equation for the determination of R(r)

$$3rR^3\frac{d^2R}{dr^2} - R^3\frac{dR}{dr} + r^3\left(\frac{dR}{dr}\right)^4 = 0 \quad a < r < b \tag{2.14}$$

Now for the pressurized cylinder considered, the prescribed boundary conditions are

$$\tau_{RR} = -p \text{ at } r = a, \quad \tau_{RR} = 0 \text{ at } r = b. \tag{2.15}$$

3.0 Solution of Boundary value Problem

We are now set to solve the boundary value problem

$$\begin{cases} 3rR^3\ddot{R} - R^3\dot{R} + r^3\dot{R}^4 = 0 \\ \tau_{RR} = -p \quad \text{at } r = a \\ \tau_{RR} = 0 \quad \text{at } r = b \end{cases}$$

When we use equation (2.11) we see that the boundary conditions are the same as

$$R(a)\,\dot{R}^3(a) = a\,(1 + p/u)^{-1}, \tag{3.1a}$$

$$R(b)\,\dot{R}^3(b) = b. \tag{3.1b}$$

Notice that here, the boundary conditions are nonlinear.

Let us set

$$R^4(r) = g^3(r). \tag{3.2}$$

Substituting equation (3.2) in the boundary value problem and realizing that

$R = g^{3/4}$,

$\dot{R} = \frac{3}{4}g^{-1/4}\dot{g}$,

$$\ddot{R} = \frac{3}{4}\left(-\frac{1}{4}g^{-5/4}\dot{g}^2 + \ddot{g}\right),$$

we obtain

$$\begin{cases} 3rg^3\ddot{g} - \frac{3}{4}rg^2\dot{g}^2 - g^3\dot{g} + \frac{3}{4}r^3\dot{g}^4 = 0 & (3.3a) \\ \dot{g}(a) = \frac{4}{3}a^{\frac{1}{3}}\beta & (3.3b) \\ \dot{g}(b) = \frac{4}{3}b^{\frac{1}{3}} & (3.3c) \end{cases}$$

where $\beta = \left(1 + \frac{p}{\mu}\right)^{-1/3}$

In (3.3), although the equilibrium equation is still nonlinear but we now have linear boundary conditions.
A solution of (3.3) in closed-form is not readily feasible, so we seek an approximate solution in $W^{1,2}(a, b)$. The Sobolev space $W^{1,p}(a,b)$, $p \geq 1$ is characterized by absolutely continuous functions. Now let

$$g = r^{4/3}\left[\frac{c_{-2}}{-r^2} + c_0 + c_2 r^2\right], \qquad (3.4)$$

where we choose c_{-2}, c_0 and c_2 so that the boundary conditions in (3.3) are satisfies and the error ε is minimum in $W^{1,2}(a,b)$
From equations (3.3) and (3.4) we obtain the relationship between c_2, c_0 and c_2 as

$$c_{-2} = \frac{4a^2b^2(1-\beta)}{b^2 - a^2} - 5a^2b^2c_2 \qquad (3.5a)$$

$$c_0 = \frac{2\left(b^2 - a^2\beta\right)}{b^2 - a^2} - 5(b^2 + a^2)\,c_2 \qquad (3.5b)$$

Hence we can replace g with g_0 as

$$g_0 = \frac{1}{b^2 - a^2}\left[4a^2b^2(1-\beta)r^{-2/3} + 2(b^2 - a^2\beta)r^{4/3}\right] - \left[5a^2b^2r^{-2/3} + 5(b^2 + a^2)r^{4/3} - r^{10/3}\right]c_2. \qquad (3.6)$$

If we denote the error in using g_0 as g by ε (r, c_2), we may write

$$\varepsilon(r, c_2) = 3r\dot{g}_0^2\ddot{g}_0 - \frac{3}{4}rg_0^2\dot{g}_0^2 - g_0^3\dot{g}_0 + \left(\frac{3}{4}\right)^3 r^3\dot{g}_0^4 \qquad (3.7)$$

Substituting (3.6) into (3.7) we obtain

$$\varepsilon(r, c_2) = \left[3r^5\ddot{n} + r^4 n - \frac{8}{3}r^3 n\right]c_2 + \left[ar^{\frac{10}{3}}n\,\ddot{n} - 7r^{\frac{8}{3}}n\dot{n} + \frac{15}{4}r^{\frac{11}{3}}\ddot{n}^2 - \frac{4}{3}r^{\frac{5}{3}}n^2\right]c_2^2 + 0(c_2^3), \qquad (3.8)$$

where

$$n(r) = r^{10/3} - 5a^2b^2r^{-2/3} - 5(b^2 + a^2)r^{4/3} \qquad (3.9)$$

The minimizing condition in the space $W^{(1,2)}$ (a, b) is given by

$$\frac{\partial}{\partial c_2}\left\|\varepsilon(\mathrm{r}, \mathrm{c}_2)\right\| = 0 \qquad (3.10)$$

where || . || denote norm in $W^{1,2}$ (a, b)
After some computations to O (c_2^3) we obtain

$$c_2 = \frac{-1}{(b^2 - a^2)} \qquad (3.11)$$

Therefore

$$g(r) = \frac{r^{-2/3}}{(b^2-a^2)}\left[a^2b^2(9-4\beta)+(7b^2+a^2(5-2\beta))\,r^2-r^4\right] \qquad (3.12)$$

so that

$$R = g^{3/4} = \frac{r^{-1/2}}{(b^2-a^2)^{3/4}}\left[a^2b^2(9-4\beta)+(7b^2+a^2(5-2\beta)\,r^2-r^4\right]^{3/4} \qquad (3.13)$$

The stress component τ_{RR}, is seen to be

$$\tau_{RR} = \mu\left[1-\frac{r}{\left(\frac{3}{4}\right)^3\dot{g}^3}\right]$$

$$= \mu\left[1-\frac{(b^2-a^2)^3 r^{8/3}}{\left(\frac{4}{3}\right)^3\left[\frac{4}{3}(7b^2+a^2(5-2\beta)\,r^2-\frac{2}{3}a^2b^2(9-4\beta)-\frac{10}{3}r^4\right]^3}\right] \qquad (3.14)$$

4.0 Summary and Conclusion

Chung *et al* [2] tried to use the technique of Abeyaratne and Horgan [1] to find a solution to this problem of pressurized cylinder, unfortunately the solution they obtained was in parametric form. Worse still was that the solution they obtained for the displacement R was in form of the ratio of deformed and undeformed radii R : r which led to a set of four highly non linear simultaneous equations for the determination of the parameters and the constants of integration. Even at that, the parameters could only be determined at the surfaces r = a and r = b.

In this paper we have presented a solution that is explicit. Equation (3.13) gives R, the deformed radius as an explicit function of r, the undeformed radius thereby permitting determination of stresses and displacements at any desired section of the cylinder. It is also easy to check the effect of increased pressure p through β on the deformed radius R. Application of the method used in this paper to the linearized problem gives the already known results of Timoshenko and Goodier [7].

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