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Stress and Displacement of an Infinite Layer under Anti-Plane Strain

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Abstract

A homogenous isotropic infinite elastic layer occupying the region $-\infty < x < \infty$, $0 \leq y \leq a$ under anti-plane strain is investigated. The problem is formulated using the Mellin transform and solved by conformal mappings. A closed form solution for the displacement everywhere in the layer is obtained from which we determine the stress field and the effects the loading caused on the material was ascertained. Three outstanding cases namely $y = a$, $y = \frac{a}{2}$ and $y = 0$, are considered.

1.0 Introduction

Some observations regarding the stress field near the point of a crack has been carried out by Williams [1]. Reissner [2] studied stress and Displacements of spherical shells.

In this paper, we investigate a homogeneous isotropic infinite elastic layer of height a in a two dimensional strip subjected to anti-plane stresses of magnitude T .

It consists of two parallel line ABC and DEF where C and D are at $-\infty$ while A and F are at $+\infty$. The loadings are prescribed on BC and DE where the strip $y = a$ and $y = 0$ respectively.

(1)

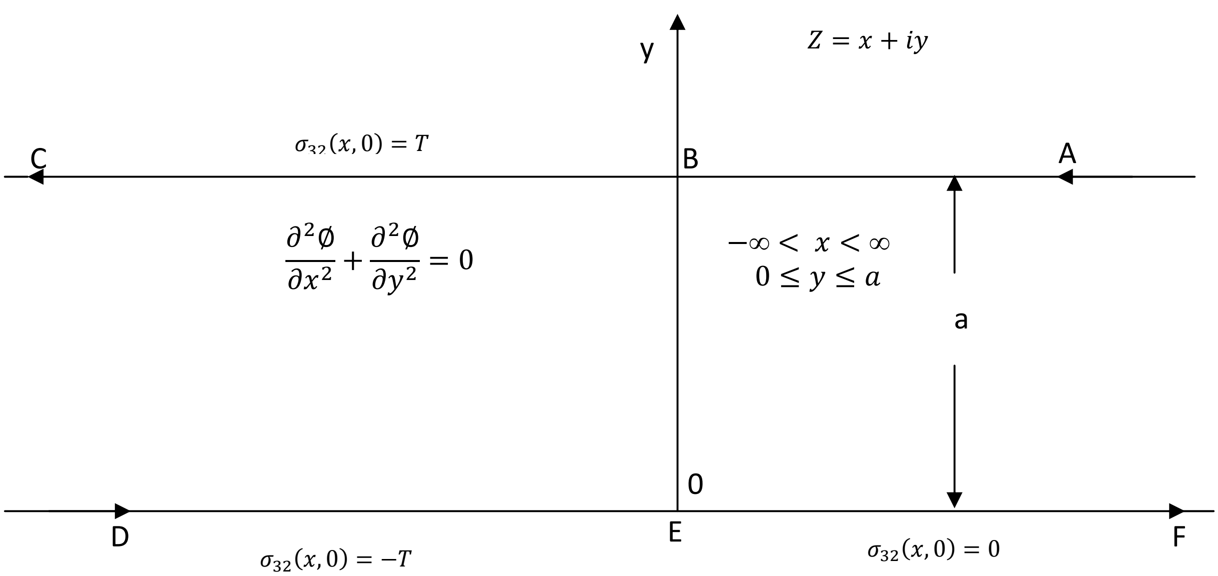

Fig 1 Geometry Of The Plane That Forms The Layer

1.1 A Transformation Of Plane Representing The Layer

Because of the nature of the boundary conditions, elementary methods of solutions such as methods of separation of variables cannot be applied. Therefore, the region of analysis is transformed to a region where integral transform can be applied.

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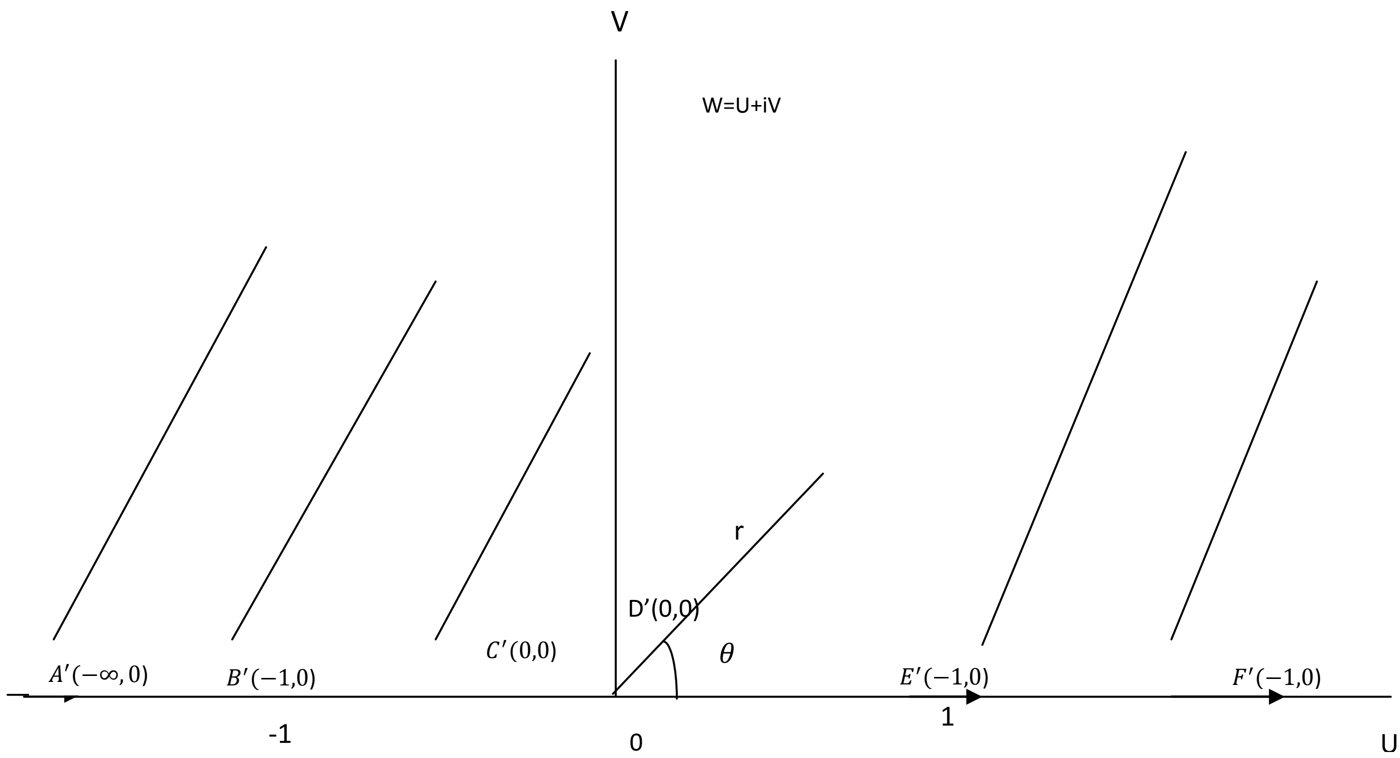

Fig 2. The Layer Transformed To An Upper Half Plane (2)

The Conformal Transformation

Puttng

$$w(z) = e^{\frac{\pi z}{a}}, \qquad z = x + iy \tag{1}$$

maps the shaded plane region in Fig 1 onto the upper half w-plane of Fig 2

Under the given special loading conditions the governing field equations of linear elasticity reduces to the following Laplace equations

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) u_3(u, v) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) u(r, \theta) = 0 \tag{2}$$

$$-1 < r < 1, \;\; 0 \leq \theta \leq \pi.$$

The non-zero stress components are given by the relation

$$\sigma_{\theta z}(r, \theta) = \frac{\mu}{r}\frac{\partial}{\partial \theta} u_3(r, \theta), \quad \sigma_{rz}(r, \theta) = \mu \frac{\partial}{\partial r} u_3(r, \theta) \tag{3}$$

Where μ denotes the shear modulus of elasticity.

The boundary conditions are

$$\frac{\partial}{\partial \theta} u_3(r, 0) = -\frac{aT}{\mu\pi}, \qquad o \leq r < 1 \tag{4}$$

$$= 0, \qquad r > 1$$

$$\frac{\partial}{\partial \theta} u_3(r, \pi) = -\frac{aT}{\mu\pi}, \qquad o \leq r \leq 1 \tag{5}$$

$$= 0, \qquad r > 1$$

The asymptotic behavior of the stresses are

$$\sigma\theta z, \sigma r z = \begin{cases} 0\left(r^{\lambda-1}\right) & as \quad r \to 0 \\ 0\left(r^{\lambda-1}\right) & as \quad r \to \infty \end{cases} \tag{7}$$

The Mellin integral transform of $u_3(r,\theta)$, is defined as

$$\bar{u}_3(s,\theta) = \int_0^\infty u_3(r,\theta) r^{s-1} dr \qquad (-1 < Res < 1) \ [3,4] \qquad (8)$$

Applying the Mellin transform to (2), (4) and (5) yields

$$\left(\frac{d^2}{d\theta^2} + s^2\right)\bar{u}_3(s,\theta) = 0 \qquad -1 < Res < 1 \qquad (9)$$

$$\frac{\partial \bar{u}_3}{\partial\theta}(s,0) = -\frac{aT}{\mu\pi s} \qquad (10)$$

$$\frac{\partial \bar{u}_3}{\partial\theta}(s,\pi) = -\frac{aT}{\mu\pi s} \qquad (11)$$

Solution of (9) is considered in the form

$$\bar{u}_3(s,\theta) = A(s)cos\theta s + B(s)sin\theta s \qquad (12)$$

Therefore (10), (11) and (12) yields

$$-s\, A(s)sin0s + sB(s)0s = -\frac{aT}{\mu\pi s} \qquad (13)$$

$$-sA(s)sin\pi s + sB(s)cos\pi s = -\frac{aT}{\mu\pi s} \qquad (14)$$

Consequently,

$$\bar{u}_3(s,\theta) = \frac{aT}{\mu\pi}\left[\frac{cos\theta s - \cos(\pi-\theta)s}{s^2 sin\pi s}\right] \qquad (15)$$

The displacement sought for is given by the inverse of the Mellin transform given by

$$u_3(r,\theta) = \int_{c-i\infty}^{c+i\infty} u_3(s,\theta) r^{-s} ds \qquad (16)$$

The Bromwich integral in (16) can be solved by method of residue. The integral has simple poles at S $= \pm n,\ n = 1, 2, 3, \ldots\ldots$ and double poles at $S = 0$

Therefore, the solution of the problem is

$$\bar{u}_3(s,\theta) = \frac{aT}{\mu\pi}\left[\frac{\pi}{2} - \theta + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(cosn\theta - \cos(\pi-\theta))\right] r^{\circ} \qquad (17)$$

$$r < 1, \qquad 0 \le \theta \le \pi$$

$$= -\frac{aT}{\mu\pi}\left[\frac{1}{n}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(cosn\theta - \cos(\pi-\theta))\right] r^{-n}$$

$$r > 1, \qquad 0 \le \theta \le \pi$$

Polar co-ordinates $u_3(r,\theta)$ obtained in (17) was converted back to its original Cartesian co-ordinate $u_3(x,y)$

$$u_3(x,y) = \begin{cases} \frac{aT}{\mu\pi}\left[\frac{\pi}{2}y + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(cosn\frac{\pi y}{a} - \cos n\pi(1-\frac{y}{a})\right)\right] e^{n\frac{\pi x}{a}},\ e^{\frac{\pi x}{a}} < 1, \\ \qquad 0 \le y \le a \\ \frac{aT}{\mu\pi}\left[\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(cosn\frac{\pi y}{a} - \cos n\pi(1-\frac{y}{a})\right)\right] e^{-n\frac{\pi x}{a}},\ e^{\frac{\pi x}{a}} > 1, \\ \qquad 0 \le y \le a \end{cases} \qquad (18)$$

Three outstanding cases namely $y = a, y = \frac{a}{2}$ and $y = 0$ are considered. These cases yield known result for $e^{\frac{\pi x}{a}} < 1$ and $e^{\frac{\pi x}{a}} > 1$

2.1 ANALYSIS OF THE RESULTS

Analysis of the displacement and stresses for $0 \le x \le \infty$ and $0 \le y \le a$

1) For $e^{\frac{\pi x}{a}} < 1$

At $y = 0$ the displacement is

$$u_3(x,0) = \frac{aT}{\mu\pi}\left[\frac{\pi}{2} - \frac{\pi}{\pi}.a + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(cosn\frac{\pi y}{a} - \cos n\pi(1-\frac{y}{a})\right)\right] e^{n\frac{\pi x}{a}},$$

$$u_3(x,0) = \frac{aT}{\mu\pi}\left[\frac{\pi}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2}\right)\right] e^{n\frac{\pi x}{a}}, \qquad (19)$$

At $y = a$, the displacement is

$$u_3(x,0) = \frac{aT}{\mu\pi}\left[\frac{\pi}{2} - \frac{\pi}{\pi}.a + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(cosn\frac{n\pi a}{a} - \cos n\pi(1-\frac{a}{a})\right)\right]e^{n\frac{\pi x}{a}},$$

$$u_3(x,0) = \frac{aT}{\mu\pi}\left[-\frac{\pi}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2}\right)\right]e^{n\frac{\pi x}{a}}, \qquad (20)$$

At $y = \frac{a}{2}$, displacement is

$$u_3(x,0) = \frac{aT}{\mu\pi}\left[\frac{\pi}{2} - \frac{\pi}{\pi}.\frac{a}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(cosn\frac{n\pi}{a}.\frac{a}{2} - \cos n(\pi - \frac{\pi}{a}.\frac{a}{2})\right)\right]e^{n\frac{\pi x}{a}},$$

$$u_3\left(x,\frac{a}{2}\right) = 0 \qquad (21)$$

Let $y = \frac{a}{n}\left(n - \frac{1}{2}\right)$, the displacement the becomes

$$u_3\left(x,\frac{a}{n}(n-\frac{1}{2})\right) = \frac{aT}{\mu\pi}\left[\frac{\pi}{2} - \frac{\pi}{a}.\frac{a}{n}\left(n-\frac{1}{2}\right) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(cosn\frac{n\pi}{a}.\frac{a}{n}\left(n-\frac{1}{2}\right) - \cos n\pi(1-\frac{1}{n}\left(n-\frac{1}{2}\right))\right)\right]e^{n\frac{\pi x}{a}},$$

$$= \frac{aT}{\mu\pi}\left[\frac{\pi}{2}.\frac{\pi}{n}\left(n-\frac{1}{2}\right) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(cosn\pi.\left(n-\frac{1}{2}\right) - \cos n(\pi - \frac{\pi}{n}\left(n-\frac{1}{2}\right))\right)\right]e^{n\frac{\pi x}{a}},$$

$$= \frac{aT}{\mu\pi}\left[\frac{\pi}{2} + \frac{\pi}{2n} + \frac{1}{n}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(0-0)\right]e^{n\frac{\pi x}{a}},$$

$$= \frac{aT}{\mu\pi}\pi\ \left[\frac{1}{2}\left(\frac{1}{n} - 1\right)\right]e^{n\frac{\pi x}{a}}, \quad e^{n\frac{\pi x}{a}}, < 1 \quad n = 1,2,3 \ldots \ldots$$

$$u_3\left(x,\frac{a}{n}\left(n-\frac{1}{2}\right)\right) = \frac{1}{2}\frac{aT}{\mu\pi}\left(\frac{1}{n} - 1\right)e^{n\frac{\pi x}{a}}, \quad n = 1,2,3 \ldots.$$

For $e^{n\frac{\pi x}{a}}, > 1$

At $y = 0$ the displacement is

$$u_3(x,0) = \frac{aT}{\mu\pi}\left[\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(cosn(0) - \cos n(\pi - 0))\right]e^{-n\frac{\pi x}{a}}$$

$$= \frac{aT}{\mu\pi}\left[\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(1-(-1)^n)\right]e^{-n\frac{\pi x}{a}}$$

$$= \frac{aT}{\mu\pi}\left[\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\right]e^{-n\frac{\pi x}{a}}$$

$$= \frac{aT}{\mu\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]e^{-n\frac{\pi x}{a}}$$

$$= \frac{2aT}{\mu\pi^2}\sum_{n=1}^{\infty}\left[\frac{e^{-(2n-1)\frac{\pi x}{a}}}{(2n-1)^2}\right] \qquad (22)$$

At $y = a$ the displacement is

$$u_3(x,a) = \frac{aT}{\mu\pi}\left[\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(cosn(\frac{n\pi a}{a}) - cosn(\pi - \frac{\pi}{a}a)\right)\right]e^{-n\frac{\pi x}{a}}$$

$$= \frac{aT}{\mu\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}[(-1)^n - 1]e^{-n\frac{\pi x}{a}}$$

$$= \frac{aT}{\mu\pi^2}\sum_{n=1}^{\infty}\left(\frac{1-(-1)}{n^2}\right)^n e^{-n\frac{\pi x}{a}}$$

$$= \frac{-2aT}{\mu\pi^2}\sum_{n=1}^{\infty}\left[\frac{e^{-(2n-1)\frac{\pi x}{a}}}{(2n-1)^2}\right] \qquad (23)$$

At $y = \frac{a}{n}$ the displacement is

$$u_3\left(x,\frac{a}{2}\right) = \frac{aT}{\mu\pi^2}\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(cos\frac{n\pi}{2} - cos\frac{\pi n}{2}\right)\right]e^{-n\frac{\pi x}{a}} = 0 \qquad (24)$$

At $y = \frac{a}{n}(n-\frac{1}{2})$ the displacement is

$$u_3\left(x, \frac{a}{n}(n-\frac{1}{2})\right) = \frac{aT}{\mu\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(cos\frac{n\pi}{a}.\frac{a}{n}\left(n-\frac{1}{2}\right) - cosn(\pi - \frac{\pi}{a}.\frac{a}{n}(n-\frac{1}{2})\right)e^{-n\frac{\pi x}{a}}$$

$$= \frac{aT}{\mu\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(cos\pi\left(n-\frac{1}{2}\right) - cosn(\pi - .\frac{\pi}{n}(n-\frac{1}{2})\right)e^{-n\frac{\pi x}{a}}$$

$$= \frac{aT}{\mu\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(0-0))\,e^{-n\frac{\pi x}{a}}$$

$$= 0 \qquad (25)$$

Plotting the graph of $\frac{\mu}{aT}\,u_3\left(x, a(n-\frac{1}{2n})\right) = -\frac{1}{2}(1-\frac{1}{n})e^{\frac{n\pi x}{a}}$

Where $y = -\frac{1}{2}\left(1-\frac{1}{n}\right)e^{n\pi x}$ $\quad n \neq 1$ i.e. $n = 2,3,4, \dots$..

We have Fig 3

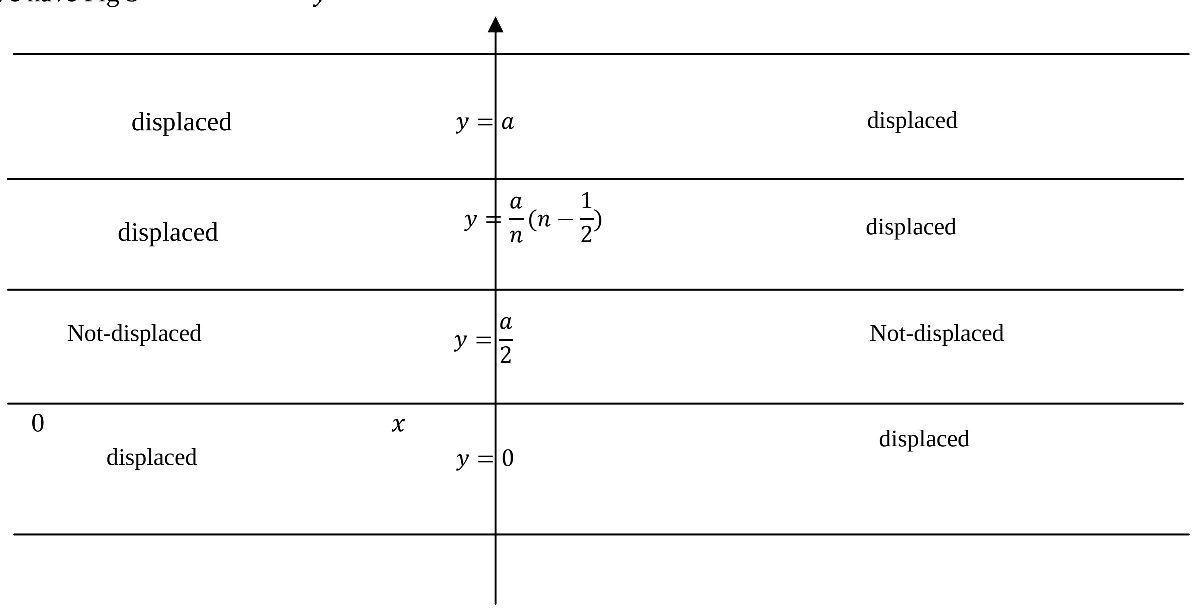

Fig 3 Layer showing displaced and non-displaced regions

General stress states are

$$\sigma_{3j}(x,y) = \frac{\mu du_3}{dx_j}(x,y), \quad j = 1,2 \dots.$$

At $y = 0$, the stress is

$$\mu\frac{du}{dy}(x,0) = \frac{aT}{\pi}\left[-\frac{\pi}{a} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{n\pi}{a}\frac{(-1)^n}{n^2}(sin\,0 - \sin n\pi)\right]e^{n\frac{\pi x}{a}},$$

$$= \frac{aT}{\pi}\left[-\frac{\pi}{a} + \frac{1}{a}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(0-0)\right]e^{n\frac{\pi x}{a}},$$

$$= \frac{aT}{\pi}\left[-\frac{\pi}{a}\right]e^{n\frac{\pi x}{a}}, \qquad n = 1,2,3$$

$$\sigma_{32}(x,0) = -Te^{n\frac{\pi x}{a}} \qquad n = 1,2,3 \dots \qquad (26)$$

At $y = a,$ the stress states is

$$\mu\frac{du_3}{dy}(x,a) = \frac{aT}{\pi}\left[-\frac{\pi}{a} + \frac{1}{a}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(-sin\,\frac{\pi}{a}.a + \sin n\left(\pi - \frac{\pi}{a}.a\right)\right)\right]e^{n\frac{\pi x}{a}},$$

$$= \frac{aT}{\pi}\left[-\frac{\pi}{a}\right]e^{n\frac{\pi x}{a}},$$

$$\sigma_{32}(x,a) = -Te^{n\frac{\pi x}{a}} \qquad (27)$$

At $y = \frac{a}{2}$ the stress states becomes

$$\mu\frac{du_3}{dy}\left(x,\frac{a}{2}\right)=\frac{aT}{\pi}\left[-\frac{\pi}{a}+\frac{1}{a}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(-sin\ \frac{\pi}{2}.n+\sin \mathrm{n}\frac{\pi}{2}\right)\right]e^{n\frac{\pi x}{a}},$$

$$\sigma_{32}\left(x,\frac{a}{2}\right)=-Te^{n\frac{\pi x}{a}} \qquad (28)$$

At $y=\frac{a}{n}\left(n-\frac{1}{2}\right)$, the stress states is

$$\mu\frac{du_3}{dy}\left(x,\frac{a}{n}\left(n-\frac{1}{2}\right)\right)=\frac{aT}{\pi}\left[-\frac{\pi}{a}+\frac{1}{a}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(-sin\ \frac{\pi}{2}(2n-1)+\sin\frac{\pi}{2}\right)\right]e^{n\frac{\pi x}{a}},$$

$$=\frac{aT}{\pi}\left[-\frac{\pi}{a}+\frac{1}{a}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{(-1)^n}{n}+1\right)\right]e^{n\frac{\pi x}{a}}$$

$$=T\left[-1+\frac{1}{\pi}\sum_{n=1}^{\infty}\left(1+\frac{(-1)^n}{n}\right)\right]e^{n\frac{\pi x}{a}}$$

$$=T\left[-1+\frac{1}{\pi}\sum_{n=1}^{\infty}e^{2k\frac{\pi x}{a}}\right] \quad \sigma_{32}\left(x,\frac{a}{2}\right)=T\left[-1+\frac{1}{\pi}\ e^{2k\frac{\pi x}{a}}\ +\sum_{n=1}^{\infty}\frac{e^{2k\frac{\pi x}{a}}}{k}\right]$$

(29)

Similarly, for $\sigma_{3j}(x,y)=\frac{\mu du_3}{dx_j}(x,y)$ respectively.

The layer experienced stress at the upper edge where

$$\sigma_{31}(x,a)=\begin{cases} nT\left[-\frac{\pi}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2}\right)\right]e^{n\frac{\pi x}{a}} & -\infty<x<0 \quad 0\le y\le a \\ -\frac{T}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\right)e^{n\frac{\pi x}{a}} & x>1, \quad 0\le y<a \end{cases} \qquad (29)$$

The material did not experience stress at the middle where

$$\sigma_{32}\left(x,\frac{a}{2}\right)=\begin{cases}0, & e^{n\frac{\pi x}{a}}<1 \\ 0 & e^{n\frac{\pi x}{a}}>1\end{cases} \qquad (30)$$

The layer also experienced stress at the lower edge where

$$\sigma_{32}(x,0)=\begin{cases}\frac{T}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n-1}{n}\right)e^{n\frac{\pi x}{a}} & e^{n\frac{\pi x}{a}}<1 \\ -\frac{T}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n-1}{n}\right)e^{n\frac{\pi x}{a}} & e^{n\frac{\pi x}{a}}>1\end{cases}$$

3.0 Conclusion

The displacement deduced in a closed form, is then used to obtain the stress everywhere in the material. Along the middle of the layer, where $y=\frac{a}{2}$, the displacement vanishes that is

$$\left(u_3\left(x,\frac{a}{2}\right)=0, \quad -\infty<x<\infty, \quad 0\le y\le a\right),$$

The stress are $\sigma_{31}\left(x,\frac{a}{2}\right)=0$ and $\sigma_{31}\left(x,\frac{a}{2}\right)\neq 0.$ show that since the tearing stress, $\sigma_{31}\left(x,\frac{a}{2}\right)$ depends on the applied stress *T*, it follows that cracking will begin at the middle of the layer if *T* becomes very large. This is line with the result of [5,6]

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