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A Concise and Direct Classification of Rank 3 Finite Groups.

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Abstract

Consider a group G with finite order such that it acts on itself by conjugation, the paper presents a direct and simple appraoch for determing all possible groups that can be obtained if this action yields precisely three orbits, that is if it yields a group of rank 3.

1.0 Introduction

This paper considers a simple and direct approach of classifying the structure of groups whose orders are finite and are of conjugate rank 3. Several research papers and journals have been written on the structures expected of finite groups as seen in [1 - 7].

This paper considers a simple and direct approach of classifying the structure of groups whose orders are finite and are of conjugate rank 3.

Using the basic definitions as listed under preliminaries the paper considers the various conditions of conjugation and the has the various results as below.

2.0 Basic Definitions And Preliminaries Results

Definition 1

The group orbit of an element $a \in G$ is defined as

$G(a) = \{ga \in G | g \in G\}$

Where a runs over all elements of the group G [8,9] .

Definition 2

A group fixed point is an orbit consisting of a single element. Thus the stabilizer of an element say $a \in G$, consists of all permutation of elements of the group that produces a group fixed point in a [9] .

Definition 3

A group action $G \times Q \to Q$ is transitive if it possesses only a single group orbit. i.e for every pair of element a and b, there is an element $g \in G$ such that $ga = b$. When this happens Q is said to be isomorphic to the left coset G/G_a. Where G_a is the stabilizer of a in G [8] .

Definition 4

If for every two pairs of points a_1, a_2 and b_1, b_2, there is an element $g \in G$ such that $ga_i = b_i$, then the group action is called doubly transitive. Similarly, a group can be triply transitive and in general , a group action is n-transitive if every set $\{a_1, \ldots, b_n\}$ of $2n$ distinct elements as a group element g such that $ga_i = b_i$ [8].

Definition 5

Here we give a concise and more mathematical definition of the previous definitions:

Let G be a transitive permutation group on Q. G also act on $Q \times Q$ via

$$(a, \beta)g = (ag, \beta g)$$

The rank of G can be defined as the number of orbits of G on $Q \times Q$.

Thus

$\{(a, a) | a \in Q\}$

is one orbit and G has rank 2 if and only if

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$\{(\alpha, \beta) | \alpha = \beta\}$

is an orbit, that is if and only if G is 2-transitive.

The rank of G can also be defined as the number of orbits of the stabilizer of a point.[3]

Lemma. If A is a group, A acts on itself by conjugation:

$$hx = hxh^{-1} \; for \; g, x \in A$$

Corollary. Let A act on itself by conjugation, and let $x \in A$. Then the number of

elements in each conjugacy class divides the order of A.

3.0 The Proof of the Main Result

We here consider a finite group G that acts on itself by conjugation, and determine all possible groups that can be obtained if this action yields precisely a group of rank 3.

To find such groups we just need to consider the conjugacy classes as below:
Obviously the identity forms its own conjugacy classes.
We have two other conjugacy classes of of size a and b respectively to find.
Now recall that a conjugacy class divides the order of the group.
Thus $a|1 + a + b$ and thus $a|1 + b$ and similarly $b|1 + a$.

So if $a = b$, then $a = b = 1$ and we get $Z/3$

Otherwise with out loss of generality $a < b$. But $b|1 + a$ means $b \leq 1 + a$, hence $b = 1 + a$.

Finally $a|1 + b = 2 + a$ gives $a = 1$ or 2, yielding $|G| = 4$ or 6 respectively.

In the former case, G is abelian, giving 4 conjugacy classes a contradiction.

So $|G| = 6$.

Of course G must be nonabelian, which means $G = S_3$.

A simple check confirms that S_3 has three conjugacy classes as we have shown above.

Thus the finite groups with three conjugacy classes are exactly $Z/3$ and S_3.

Conclusion and Future Works

The above results have yielded concise and simple approach of classifying groups of lower ranks and as such have opened a path for considering other groups with similar structures.

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