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Power Series Solution Method for Riccati Equation

Disu A . Babatunde , Ishola C. Yemisi and Olorunnishola Toyin

School of Science and Technology,
National Open University of Nigeria
P.M.B.80067, Victoria Island, Lagos, Nigeria

Abstract

In this paper, power series solution method (PSSM) is applied to solve Riccati equations. The equations under consideration are of variable coefficient form.Comparisons with exact solutions show that (PSSM) is a powerful method for the solution of nonlinear equations.The method gives realistic series of solutions that converge rapidly.

Keywords: Power series solution method, Riccati equation

1.0 Introduction

A Riccati equation is an ordinary differential equation that is quadratic in the unknown function. The Riccati differential equation is named after the Italian nobleman Count Jacopo Francesco Riccati (1676-1754). The book of Reid [1] contains the fundamental theories of Riccati equation with applications to random processes,optimal control and diffusion problems. Beside important engineering and science applications that today are known as classically proved, such as stochastic realization theory, optimal control, robust stabilization, and network synthesis,the newer applications which include such areas as financial mathematics [2, 3].

The importance of Riccati equations in many fields attract the attention of many authors.The solution of this equation can be reached using classical numerical methods such as the forward Euler method and Runge-Kutta method. An unconditionally stable scheme was presented by Dubois and Saidi [4]. Bahnasawi et al. [5] presented the usage of Adomian decomposition method (ADM) to solve the nonlinear Riccati in an analytic form. Very recently, Tan and Abbasbandy [6] employed the analytic technique called Homotopy Analysis Method (HAM) to solve a quadratic Riccati equation.Abbasbandy [7] presented a new application of He's variational method for a general Riccati equation using Adomian polynomials.Batiha et al[8] presented application variational iteration method to solve a general Riccatic equation .Disu et al[9] have presented application of Adomian decomposition method (ADM) to a general Ricattic equation.Biazar and Eslami [10] applied differntial transform method (DTM) for quadratic riccatic equation.

Power Series Solution (PSS) Method (PSSM) has been found as an effective method for solving Linear Differential Equations (LDE).The study of the PSSM was limited to LDE. Until recently,Nuran and Mustafa [11] presented PSS of Nonlinear First Order Differential Equation System.Liao and and Tan [12] proposed a general approach to obtain series solutions of NonLinear Differential equations (NLDEs). Lopez-Sandoval and Mello [13] presented PSS of Nonlinear Partial Differential equations (NLPDEs) from Mathematical Physics.

In this paper, we present approximate solutions for Riccati equations using Power Series Solution Method

$$\frac{dy}{dt} = Q(t)y + R(t)y^2 + p(t) \quad ; \quad y(0) = G(t) \qquad (1)$$

where $Q(t)$, $R(t)$, $P(t)$, $G(t)$ are known as scalar functions.

In particular, we give 3 examples of Riccati equations which include variable coefficient forms. Numerical comparisons between exact solutions on these equations are given.

Corresponding author ***A. Disu,*** E-mail: modelab4real@yahoo.com-, Tel. +234-8077138381

2.0 The Power Series Solution Method

The power series (in powers of $t-t_0$) is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (t-t_0)^n = a_0 + a_1(t-t_0) + a_2(t-t_0)^2 + a_3(t-t_0)^3 + \cdots \tag{2}$$

where $a_0, a_1, a_2, a_3, \cdots$ are constants called the coefficients of the series and t_0 is a constant called the center of the series and t is a variable.

If in particular $t_0 = 0$,we obtain a power series in powers of t .

$$\sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n + \cdots \tag{3}$$

Series (3) is said to converge absolutely if $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| < 1$,

where $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ .Thus if the given t lies in this interval i.e $|x| < R$, then the corresponding sum of the series is a function of t is given by $y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n + \cdots = \sum_{n=0}^{\infty} a_n t^n$ (4)

Also, $y'(t) = a_1 + a_2 t + a_3 t^2 + \cdots + a_n t^{n-1} + \cdots = \sum_{n=1}^{\infty} n a_n t^{n-1}$ (5)

2.0 Numerical Examples

Example 1 Consider the following equation [5]

$$\frac{dy}{dt} = y^2(t) \quad ; \quad y(0) = 1 \tag{6}$$

The exact solution was found by [5] as

$$y(t) = \frac{1}{1-t} \tag{7}$$

The series solutions are

$$y(t) = \sum_{n=0}^{\infty} a_n t^n \;\; ; \;\; y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1} \tag{8}$$

substituting (8) in (6),we obtain from the initial condition, $a_0 = 0 \Rightarrow a_1 = 1$

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = \left(\sum_{n=0}^{\infty} a_n t^n\right)^2 \tag{9}$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} t^n - \sum_{n=0}^{\infty} t^n \sum_{j=1}^{n} a_{n-j} a_j = 0 \tag{10}$$

$$a_1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} t^n + a_0^{\,2} + \sum_{n=1}^{\infty} t^n \sum_{j=0}^{n} a_{n-j} a_j = 0 \tag{11}$$

$$a_1 = a_0^{\,2} ; \quad a_{n+1} = \frac{-\sum_{j}^{n} a_{n-j} a_j}{n+1} \tag{12}$$

from the initial condition,we obtain $a_0 = 1, \Rightarrow a_1 = 1$ (13)

$$n = 1; \quad a_2 = \frac{-2a_0 a_1}{2} = 1 \tag{14}$$

$$n = 2; \quad a_3 = \frac{-(2a_0 a_1 + a_1^{\,2})}{3} = 1 \tag{15}$$

$$n = 3; \quad a_4 = \frac{-(2a_0 a_3 + 2a_1 a_2)}{4} = 1 \tag{16}$$

$$n = 4; \quad a_5 = \frac{-(2a_0 a_4 + 2a_1 a_3 + a_2^{\,2})}{5} = 1 \tag{17}$$

$$n = 5; \quad a_6 = \frac{-(2a_0 a_5 + 2a_1 a_4 + 2a_2 a_3)}{6} = 1 \tag{18}$$

$$n = 6; \quad a_7 = \frac{-(2a_0 a_6 + 2a_1 a_5 + 2a_2 a_4 + a_3^{\,2})}{7} = 1 \tag{19}$$

$$n = 7; \quad a_8 = \frac{-(2a_0 a_7 + 2a_1 a_6 + 2a_2 a_5 + 2a_3 a_4)}{8} = 1 \tag{20}$$

$$n = 8; \quad a_9 = \frac{-(2a_0 a_8 + 2a_1 a_7 + 2a_2 a_6 + 2a_3 a_5 + a_4^{\,4})}{9} = 1 \tag{21}$$

... Etc

Therefore,the form of the solution can be written as

$$y(t) = \sum_{n=0}^{\infty} a_n t^n = 1 + t + t^2 + t^3 + t^4 + t^5 + t^6 + t^7 + t^8 + t^9 \tag{22}$$

Example 2 Consider the following equation [10]

$$\frac{dy}{dt} = -y^2(t) + 1 \,, \; y(0) = 0 \tag{23}$$

The exact solution was found by [10] as

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1} \tag{24}$$

substituting (8) in (22),we obtain from the initial condition, $a_0 = 0, \Rightarrow a_1 = 1$

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = -\left(\sum_{n=0}^{\infty} a_n t^n\right)^2 + 1 \tag{25}$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} t^n + \sum_{n=0}^{\infty} t^n \sum_{j=1}^{n} a_{n-j} a_j = 1 \tag{26}$$

$$a_1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} t^n + a_0^{\,2} + \sum_{n=1}^{\infty} t^n \sum_{j=0}^{n} a_{n-j} a_j = 1 \tag{27}$$

$$a_1 = 1 - a_0^{\,2}, \quad a_{n+1} = \frac{-\sum_{j}^{n} a_{n-j} a_j}{n+1} \tag{28}$$

from the initial condition,we obtain $a_0 = 0, \Rightarrow a_1 = 1$ (29)

$$n = 1; \quad a_2 = \frac{-2a_0 a_1}{2} = 0 \tag{30}$$

$$n=2\,;\quad a_3=\frac{-(2a_0a_1+a_1^{\,2})}{3}=-\frac{1}{3} \tag{31}$$

$$n=3\,;\quad a_4=\frac{-(2a_0a_3+2a_1a_2)}{4}=0 \tag{32}$$

$$n=4\,;\quad a_5=\frac{-(2a_0a_4+2a_1a_3+a_2^{\,2})}{5}=\frac{2}{15} \tag{33}$$

$$n=5\,;\quad a_6=\frac{-(2a_0a_5+2a_1a_4+2a_2a_3)}{6}=0 \tag{34}$$

$$n=6\,;\quad a_7=\frac{-(2a_0a_6+2a_1a_5+2a_2a_4+a_3^{\,2}\ )}{7}=-\frac{17}{315} \tag{35}$$

$$n=7\,;\quad a_8=\frac{-(2a_0a_7+2a_1a_6+2a_2a_5+2a_3a_4)}{8}=0 \tag{36}$$

$$n=8\,;\quad a_9=\frac{-(2a_0a_8+2a_1a_7+2a_2a_6+2a_3a_5+a_4^{\,4})}{9}=\frac{62}{2835} \tag{37}$$

... Etc

Therefore,the form of the solution can be written as

$$y(t)=\sum_{n=0}^{\infty}a_nt^n=t-\frac{t^3}{3}+\frac{2t^5}{15}-\frac{17t^7}{315}+\frac{62t^9}{283} \tag{38}$$

Example 3 Consider the following equation[10]

$$\frac{dy}{dt}=2y(t)-y^2(t)+1\quad ,y(0)=0 \tag{39}$$

The exact solution was found by [10] as

$$y(t)=1+\sqrt{2}\tanh\left(\sqrt{2t}+\frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right) \tag{40}$$

substituting (8) in (39),we obtain

$$\sum_{n=1}^{\infty}na_nt^{n-1}+\left(\sum_{n=0}^{\infty}a_nt^n\right)^2-2\sum_{n=0}^{\infty}a_nt^n=1 \tag{41}$$

$$a_1+\sum_{n=1}^{\infty}(n+1)a_{n+1}t^n+a_0^{\,2}+\sum_{n=1}^{\infty}t^n\sum_{j=0}^{n}a_{n-j}a_j-2a_0-2\sum_{n=1}^{\infty}a_nt^n=1 \tag{42}$$

$$a_1=1+2a_0-a_0^{\,2}\,,\ a_{n+1}=\frac{-\sum_{j}^{n}a_{n-j}a_j+2a_n}{n+1} \tag{43}$$

From the the initial condition, $a_0\ =0\,,\ \Rightarrow a_1=1$ (44)

$$n=1,\quad a_2=\frac{-2a_0a_1+2a_1}{2}=1 \tag{45}$$

$$n=2\,,\ a_3=\frac{-(2a_0a_1+a_1^{\,2})+2a_2}{3}=\frac{1}{3} \tag{46}$$

$$n=3\,,\quad a_4=\frac{-(2a_0a_3+2a_1a_2)+2a_3}{4}=-\frac{1}{3} \tag{47}$$

$$n=4,\quad a_5=\frac{-(2a_0a_4+2a_1a_3+a_2^{\,2})+2a_4}{5}=-\frac{7}{15} \tag{48}$$

$$n=5,\quad a_6=\frac{-(2a_0a_5+2a_1a_4+2a_2a_3)+2a_5}{6}=-\frac{7}{45} \tag{49}$$

$$n=6,\quad a_7=\frac{-(2a_0a_6+2a_1a_5+2a_2a_4+a_3^{\,2})+2a_6}{7}=\frac{53}{315} \tag{50}$$

...Etc

Therefore, the form of the solution can be written as

$$y(t)=\sum_{n=0}^{\infty}a_nt^n=t+t^2+\frac{t^3}{3}-\frac{t^4}{3}-\frac{7t^5}{15}+\frac{53t^7}{315}+... \tag{51}$$

4.0 Numerical results and discussion

We now obtain numerical solutions of Riccati differential equations. Table 1 shows comparison between the PSSM and the exact solution for example 1. Table 2 shows comparison between the PSSM and the exact solution for example 2. Table 3 shows comparison between the PSSM and the exact solution for example 3.

Table 1: Numerical comparisons for example 1

T	Exact Solution	PSSM	Absolute Error
0.01	1.01	1.01010101	0.00010101
0.02	1.02	1.020408163	0.000408163
0.03	1.03	1.030927835	0.000927835
0.04	1.04	1.041666667	0.001666667
0.05	1.05	1.052631579	0.002631579
0.06	1.06	1.063829787	0.003829787
0.07	1.07	1.075268817	0.005268817
0.08	1.08	1.086956522	0.006956522
0.09	1.09	1.098901099	0.008901099
0.1	1.1	1.111111111	0.011111111

Table 2: Numerical comparisons for example 2

T	Exact solution	PSSM	Absolute Error
0.1	0.099667995	0.099799995	0.000132
0.2	0.197353203	0.19839932	0.001046117
0.3	0.291316124	0.294588628	0.003272504
0.4	0.379948962	0.387117311	0.007168349
0.5	0.462117157	0.474621087	0.01250393
0.6	0.537049567	0.555509628	0.018460061
0.7	0.604367777	0.627837995	0.023470218
0.8	0.66403677	0.68921731	0.02518054
0.9	0.71629787	0.736859839	0.020561969
1	0.761594156	0.767901235	0.006307079

Table 3: Numerical comparisons for example 3

T	Exact Solution	PSSM	Absolute Error
0.1	0.110295197	0.110295178	1.89222E-08
0.2	0.241976799	0.241974044	2.75476E-06
0.3	0.395104848	0.3950526	5.22481E-05
0.4	0.567812166	0.567384178	0.000427988
0.5	0.756014393	0.753819444	0.002194948
0.6	0.953566216	0.9452544	0.008311815
0.7	1.15294896	1.127566378	0.025382582
0 .8	1.34636355	1.280438044	0.065925506
0.9	1.526911312	1.3760694	0.150841912
1	1.68949839	1.377777778	0.311720612

Conclusion

In this paper, power series solution method is applied to solve Riccati equations.The method is applied in a direct way without using linearization, transformation,perturbation,discretisation or restrictive assumptions.It may be concluded that power series solution method is very powerful and efficient in finding the approximate solution for the class of differential equation. Therefore, it could be easily included in lectures on classical mechanics for undergraduate students.

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