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Semi-Analytic Solution of the Nonlinear Kawahara Equation.

[1]Olusola Tosin Kolebaje, [2]Obende Micheal and [1]Lateef Mustapha

[1]Department of Physics,
Adeyemi College of Education, Ondo, Nigeria.
[2]Department of Physics,
Afe Babalola University, Ado-Ekiti, Nigeria.

Abstract

In this paper, we applied the semi-analytic Variational Iteration method (VIM) and the Reduced Differential Transform method (RDTM) to the nonlinear fifth order Kawahara equation to test the efficacy of the methods. The study shows that these semi-analytic methods are efficient tools for obtaining approximate solutions to nonlinear equations with wide applications in physics and engineering. Conclusively, VIM was found in the example discussed to be better than RDTM due to its faster convergence rate and because it does not require calculating the cumbersome Adomian polynomials

Keywords: Variational Iteration method, Reduced differential transform method, Kawahara equation, nonlinear equation.Nomenclature, Symbols and Notations

1.0 Introduction

Most physical phenomena are best described by partial differential equations that are nonlinear in nature. These nonlinear partial differential equations appear in many fields such as Hydrodynamics, Engineering, Quantum field theory, Optics, Plasma physics etc. They mostly do not have exact solutions and are therefore approximated using numerical schemes [1].
Semi-analytic methods which do not involve round off errors and discretization of the variables associated with numerical schemes and are also relatively easier to implement [2]. They also converge to the exact solution of the nonlinear partial differential equation if an exact solution exists. These semi-analytic methods include the Adomian decomposition method [3], Differential transform method [4], Homotopy perturbation method [5], Variational iteration method [6], Reduced differential transform method [7] etc.
The Variational iteration method has been applied to a wide range of nonlinear problems, [8-11]. The Reduced differential transform method has been applied to various nonlinear partial differential equations. [7, 12-14].
In this paper we applied the VIM and RDTM to the nonlinear fifth order Kawahara equation given by:

$$\frac{\partial u}{\partial t}+ u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}-\frac{\partial^5 u}{\partial x^5}=0 \qquad (1)$$

Subject to the initial condition:

$$u(x,0)= \frac{105}{169}\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right) \qquad (2)$$

This equation occurs in the theory of magneto-acoustics waves in plasma [15] and in the theory of shallow water waves with surface tension [16].

2.0 Methodology

2.1 Variational Iteration Method

To explain the basic concept of variational iteration method [6], let's consider a differential equation in the form:

$$L\big(u(x,t)\big)+ N\big(u(x,t)\big) = g(x,t) \qquad (3)$$

Where L is a linear operator, N is a nonlinear operator and $g(x,t)$ is an inhomogeneous term. We construct a correction functional as follows:

$$u_{k+1}(x,t)= u_k(x,t)+ \int_0^t \lambda\{L\big(u_k(x,\zeta)\big)+N\big(\tilde{u}_k(x,\zeta)\big)-g(x,\zeta)\}d\,\zeta \qquad (4)$$

λ is a general Lagrange multiplier [6][17] which can be identified optimally via variational theory. The second term on the right is the correction and $\tilde{u}_k$ is a restricted variation i.e. $\delta\tilde{u}_k=0$ [6].
So we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The

Corresponding author: ***Kolebaje O. T.***, E-mail: Olusolakolebaje2008@gmail.com-, Tel. 2347038680315

determination of the approximations $u_k(x,t)$ $(k \geq 0)$ follows immediately. The solution to the differential equation is given by

$$u(x,t) = \lim_{k\to\infty} u_k(x,t) \tag{5}$$

2.2 REDUCED DIFFERENTIAL TRANSFORM METHOD

The basic definitions of the Reduced Differential Transform Method (RDTM) are introduced as follows (7);

If the function $u(x,t)$ is analytic and differentiable continuously with respect to time t and space x in the domain of interest, then we introduce $U_k(x)$ such that,

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0} \tag{6}$$

Where the t-dimensional spectrum function $U_k(x)$ is the transformed function of $u(x,t)$. The inverse differential transform of $U_k(x)$ is defined as follows;

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)\, t^k \tag{7}$$

Combining equation (6) and (7) we can write

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0} t^k \tag{8}$$

From the above definitions, it can be seen that the concept of the RDTM is derived from the power series expansion. According to the Reduced Differential Transform Method and Table 1 from [7], [12], [13] we can construct an iteration formula to obtain values of $U_1,\ U_2,\ U_3,\ U_4,\ U_5,\ etc.$

The first term of the transformed solution $U_0(x)$ is just the initial condition at the time $t = 0$. Also, for second order partial differential equations, the derivative of the initial condition with respect to time gives $U_1(x)$.The inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{n}$ gives approximate solution as

$$\overline{u_n}(x,t) = \sum_{k=0}^{n} U_k(x)\, t^k \tag{9}$$

where n is the order of the approximation. Therefore, the exact solution of the equation is given by;

$$u(x,t) = \lim_{n\to\infty} \overline{u_n}(x,t) \tag{10}$$

Table 1: Reduced Differential Transform Table

FUNCTIONAL FORM	TRANSFORMED FORM
$u(x,t)$	$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0}$
$u(x,t) \pm v(x,t)$	$U_k(x) \pm V_k(x)$
$\alpha u(x,t)$	$\alpha U_k(x)$(α is a constant)
$x^m t^n$	$x^m \delta(k-n)$
$x^m t^n u(x,t)$	$x^m U(k-n)$
$u(x,t)\, v(x,t)$	$\sum_{r=0}^{k} U_r(x)\, V_{k-r}(x)$
$\frac{\partial^r}{\partial t^r} u(x,t)$	$\frac{(k+r)!}{k!} U_{k+r}(x)$
$\frac{\partial}{\partial x} u(x,t)$	$\frac{\partial}{\partial x} U_k(x)$
Nonlinear Function $N(u(x,t)) = F(u(x,t))$	$N_k = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} F(U_0)\right]_{t=0}$

3.0 APPLICATION

3.1 VARIATIONAL ITERATION METHOD

Consider the Kawahara equation (1):

$$\frac{\partial u}{\partial t}+ u\frac{\partial u}{\partial x}+ \frac{\partial^3 u}{\partial x^3}- \frac{\partial^5 u}{\partial x^5}=0$$

Subject to the initial condition:

$$u(x,0)= \frac{105}{169}\text{sech}^4\left(\frac{x}{2\sqrt{13}}\right)$$

The equation can be rewritten for convenience as:

$$u_t+uu_x+u_{xxx}-u_{xxxxx}=0 \quad (11)$$

According to VIM, we can construct the correction functional as:

$$u_{k+1}(x,t)= u_k(x,t)+ \int_0^t \lambda\{u_{kt}+\tilde{u}_k\tilde{u}_{kx}+\tilde{u}_{kxxx}-\tilde{u}_{kxxxxx}\}d\,\zeta \quad (12)$$

Where $\tilde{u}_k$ is considered as a restricted variation, i.e. $\delta\tilde{u}_k=0$ and λ is the general Lagrange multiplier. Making the above correction functional stationary, we yield the stationary conditions:

$$1+\lambda=0 \qquad \lambda'=0$$

$$\lambda= -1$$

Substituting this value of the Lagrange multiplier into the correction functional gives the iteration formula

$$u_{k+1}(x,t)= u_k(x,t)- \int_0^t \{u_{kt}+u_ku_{kx}+u_{kxxx}-u_{kxxxxxx}\}d\,\zeta \quad (13)$$

We begin with the initial condition as the initial approximation u_0

$$u_0= \frac{105}{169}\text{sech}^4\left(\frac{x}{2\sqrt{13}}\right)$$

Using the iteration formula and the initial approximations, we obtain successive approximations $u_1,u_2,u_3,u_4,\ldots$ with the aid of computer algebraic system MATHEMATICA. The first few solutions are presented below.

$$u_1= \frac{105}{169}\text{sech}^4\left(\frac{x}{2\sqrt{13}}\right)+ \frac{7560t}{28561\sqrt{13}}\text{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\tanh\left(\frac{x}{2\sqrt{13}}\right)$$

3.2 REDUCED DIFFERENTIAL TRANSFORM METHOD

Recall the nonlinear Kawahara equation given by (11):

$$u_t+uu_x+u_{xxx}-u_{xxxxx}=0$$

(11) can be written in standard operator form as (14), where $L= \frac{\partial}{\partial t}$, $R= \frac{\partial^3}{\partial x^3}-\frac{\partial^5}{\partial x^5}$ and $N=u\frac{\partial}{\partial x}$ is the non-linear term.

$$L\big(u(x,t)\big)+R\big(u(x,t)\big)+N\big(u(x,t)\big)=0 \quad (14)$$

with initial condition

$$u(x,0)= \frac{105}{169}\text{sech}^4\left(\frac{x}{2\sqrt{13}}\right)$$

Using Table 1, we can transform each term of equation (14) as follows;

$$L\big(u(x,t)\big)= \frac{(k+1)!}{k!}U_{k+1} \quad (15)$$

$$R\big(u(x,t)\big)= \frac{\partial^3 U_k}{\partial x^3}-\frac{\partial^5 U_k}{\partial x^5} \quad (16)$$

The first few non-linear terms are the Adomian Polynomials [3] and are given as:

$$N_0= U_0\frac{\partial U_0}{\partial x}$$

$$N_1= U_0\frac{\partial U_1}{\partial x}+U_1\frac{\partial U_0}{\partial x}$$

$$N_2=U_0\frac{\partial U_2}{\partial x}+U_1\frac{\partial U_1}{\partial x}+U_2\frac{\partial U_0}{\partial x}$$

$$N_3= U_0\frac{\partial U_3}{\partial x}+U_1\frac{\partial U_2}{\partial x}+U_2\frac{\partial U_1}{\partial x}+U_3\frac{\partial U_0}{\partial x}$$

Substituting equation (15) and (16) into equation (14) we get the transformed form of the Kawahara equation.

$$\frac{(k+1)!}{k!}U_{k+1}= \frac{\partial^5 U_k}{\partial x^5}-\frac{\partial^3 U_k}{\partial x^3}-N_k \quad (17)$$

From the initial condition we can write

$$U_0=\frac{105}{169}\text{sech}^4\left(\frac{x}{2\sqrt{13}}\right)$$

Subsequent U_k values are obtained using the iteration formula (17) and the initial condition (2) with the first few values presented below.

$$U_1 = \frac{7560}{28561\sqrt{13}} \operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right) \tanh\left(\frac{x}{2\sqrt{13}}\right)$$
$$U_2 = \frac{68040}{62748517} \operatorname{sech}^6\left(\frac{x}{2\sqrt{13}}\right)\left[2\cosh\left(\frac{x}{2\sqrt{13}}\right) - 3\right]$$

The solution is given by the inverse differential transform in equation (9):

$$u(x,t) = \sum_{k=0}^{\infty} U_k\, t^k = U_0 + U_1 t^1 + U_2 t^2 + U_3 t^3 + U_4 t^4 + U_5 t^5 + U_6 t^6 \ldots \qquad (18)$$

The implementation of the RDTM is done with the aid of MATHEMATICA to obtain the U_k values and plug them into equation (18) to get the approximate solution.

4.0 RESULTS AND DISCUSSION

The variational iteration method and the reduced differential transform method have been applied to the nonlinear Kawahara equation with results obtained for the VIM for $k = 4$ while RDTM results are obtained for $k = 6$. The absolute differences between the VIM and RDTM solutions at selected times are presented in Table 2. From Table 2, we observe that the VIM and RDTM gave comparable results with a maximum absolute difference of order 1E-09 at $t = 0.2$ which implies that the VIM and RDTM results agree to at least 8 decimal places. Figure 1 shows the 3-D surface plot of the VIM and RDTM solutions of the nonlinear Kawahara equation. Identical plots are obtained from the VIM AND RDTM solutions.
Conclusively, the VIM is simpler to implement and more efficient than the RDTM as it converges faster than the RDTM and it does not require the calculation of the cumbersome and difficult Adomian Polynomials.

Table 2: Absolute differences between VIM and RDTM for the Kawahara equation.

|VIM – RDTM|							
x	$t = 0.05$	$t = 0.1$	$t = 0.2$	x	$t = 0.05$	$t = 0.1$	$t = 0.2$
-10	2.160E-14	6.890E-13	2.202E-11	**1**	8.936E-12	2.859E-10	9.147E-09
-9	1.550E-14	4.936E-13	1.574E-11	**2**	2.394E-12	7.671E-11	2.461E-09
-8	9.699E-14	3.115E-12	9.979E-11	**3**	4.504E-12	1.441E-10	4.607E-09
-7	2.620E-13	1.066E-11	3.457E-10	**4**	1.215E-12	3.935E-11	1.263E-09
-6	1.710E-13	2.621E-12	7.735E-11	**5**	1.473E-12	4.820E-11	1.544E-09
-5	1.571E-12	4.843E-11	1.547E-09	**6**	7.702E-14	2.180E-12	7.990E-11
-4	1.233E-12	3.922E-11	1.252E-09	**7**	4.850E-13	1.112E-11	3.472E-10
-3	4.506E-12	1.442E-10	4.619E-09	**8**	9.916E-13	4.900E-12	1.030E-10
-2	2.387E-12	7.630E-11	2.435E-09	**9**	6.690E-13	1.805E-12	5.745E-12
-1	8.939E-12	2.861E-10	9.157E-09	**10**	3.612E-13	1.370E-12	2.347E-11
0	3.997E-15	2.920E-13	1.868E-11				

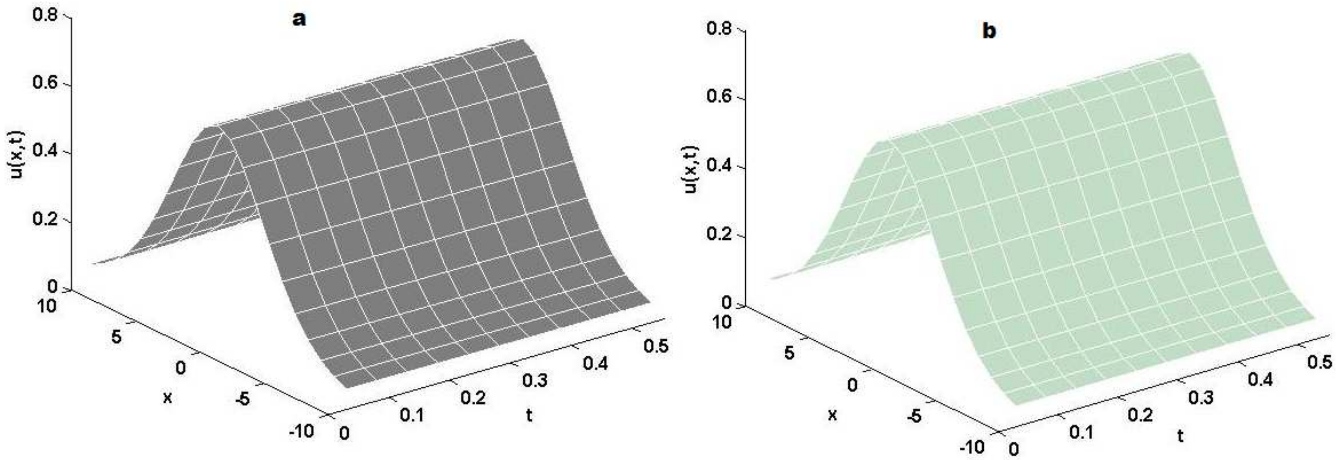

Figure 1: 3-D surface plot of solution $u(x,t)$ of the Kawahara equation with (a) VIM $(k = 4)$ (b) RDTM $(k = 6)$.

CONCLUSION

In this work, the variational iteration method and the reduced differential transform were successfully applied to obtain solutions to a nonlinear fifth order partial differential equation known as the Kawahara equation. Though VIM and the RDTM gave comparable solution, the VIM is proposed for solving other high order nonlinear partial differential equations due to its faster convergence rate and because it does not require calculating the cumbersome Adomian polynomials.

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