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Some Common Fixed Point Theorems For Weakly Compatible Mappings Satisfying A General Contractive Condition Of Integral Type

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Abstract

In this paper, some common fixed point theorems are proved for a pair of weakly compatible mapping satisfying a general contractive condition of integral type in a metric space. Our theorem has significant improvement on Jungck's fixed point theorem by employing weakly compatible maps which is more general than commutativity of maps. These results are extensions and generalizations of multitude of results in the literature, including the results of Branciari [1] and Rhoades [2].

Keywords: Jungck's fixed point theorem, common fixed point, weakly compatible maps, contractive condition of integral type, metric space.

1.0 Introduction

The study of metric fixed point theory has been researched extensively in the past decades, since fixed point theory plays a vital role in Mathematics and applied sciences, such as Optimization, Mathematical models and Economic theories. The concept of fixed point has been generalized by different authors from single map to pairs of maps, for example see ([3], [4], [5], [6]). However, the concept has also been considered in different spaces like Metric, Normed linear, Partial metric and Cone metric spaces.

The first important result on fixed points for contractive type mapping was given by Banach [7] in 1922 as stated in the following theorem: **Theorem 1.1 (Banach's Contraction Principle):** Let (X,d) be a metric space, $\delta \in (0,1)$ and $f: X \rightarrow X$ be a mapping such that for each $x, y \in X$,

$$d(fx, fy) \leq \delta d(x,y) \tag{1.1}$$

then f has a unique fixed point $p \in X$, such that for each $x \in X$, $\lim_{n\to\infty} f^n x = p.$

In 2002, Branciari [1] analyzed the existence of fixed point for mapping f defined on a complete metric space (X,d) satisfying a general contractive condition of integral type.

Theorem 1.2 (Branciari [1]): Let (X,d) be a complete metric space, $c \in (0,1)$ and $f: X \rightarrow X$ be a mapping such that for each $x, y \in X$,

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq c\int_0^{d(x,y)} \varphi(t)dt \tag{1.2}$$

where $\varphi:[0,+\infty) \rightarrow [0,+\infty)$ is a Lebesgue integrable mapping which is summable (i.e with finite integral) on each compact subset of $[0,+\infty)$, nonnegative , and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$, then f has a unique fixed point $p \in X$, such that for each $x \in X$, $\lim_{n\to\infty} f^n x = p.$

After the fine work done by Branciari [1], several research have been carried out by different authors, generalizing various contractive conditions of integral type for different contractive properties. A beautiful work has been done by Rhoades [2] extending the results of Branciari [1] by replacing condition (1.2) by the following:

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq c\int_0^{\max[d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2}]} \varphi(t)dt \ \ldots \tag{1.3}$$

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Dedication: The authors hereby dedicate this Paper to LATE PROF. R. O. AYENI.

The main aim of this paper is to generalize the fine work of Branciari [1] and Rhoades [2] by using a pair of weakly compatible maps satisfying a general contractive condition of integral type in a metric space.
We shall need the following definition to prove our results:

Definition 1.3 [6]: A point $p \in X$ is called a coincident point of a pair of self maps S,T if there exists a point q (called a point of coincidence) in X such that $q = Sp = Tp$. Self maps S and T are said to be weakly compatible if they commute at their coincidence points, that is if $Sp = Tp$ for some $p \in X$, then $STp = TSp$.

2.0 Main Results

Theorem 2.1. Let (X,d) be a metric space and $S,T : X \to X$ two self mappings of X such that $T(X) \subseteq S(X)$ satisfying

$$\int_0^{d(Tx,Ty)} \varphi(t)d(t) \le k \int_0^{d(Sx,Sy)} \varphi(t)d(t), \qquad \text{………………………………} \quad (2.1)$$

for all $x, y \in X$, where $k \in [0,1)$ and $\varphi : R^+ \to R^+$ is a Lebesgue integrable mapping, which is summable, nonnegative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)d(t) > 0.$ Suppose $S(X)$ is complete and S and T are weakly compatible. Then S and T have a unique common fixed point p in X.

Proof:

Let $x_0 \in X$, choose $x_1 \in X$ and define $Sx_1 = Tx_0$ since $T(X) \subseteq S(X)$.

We define a sequence $\{y_n\}$ in X such that

$$y_n = Sx_{n+1} = Tx_n, \text{ for } n = 0,1,2,...$$

From 2.1, for each $n \ge 1$, we have

$$\begin{aligned}\int_0^{d(y_n,y_{n+1})} \varphi(t)dt &= \int_0^{d(Tx_n,Tx_{n+1})} \varphi(t)dt \\ &\le k\int_0^{d(Sx_n,Sx_{n+1})} \varphi(t)dt, \\ &\le k^2\int_0^{d(Sx_{n-1},Sx_n)} \varphi(t)dt, \\ &\le k^n\int_0^{d(Sx_1,Sx_2)} \varphi(t)dt, \\ &\le k^n\int_0^{d(y_0,y_1)} \varphi(t)dt. \qquad \text{……………. ………} \quad (2.2)\end{aligned}$$

Since $k \in [0,1)$,

Then $\le \int_0^{d(y_n,y_{n+1})} \varphi(t)dt \to 0$ as $n \to \infty$,

Therefore, by the condition in theorem 2.1, we have $d(y_n, y_{n+1}) \to 0$ as $n \to \infty$.

Next, we show that $\{y_n\}$ is a Cauchy sequence, that is, we show that $d(y_{m(q)-1}, y_{n(q)-1}) \le \varepsilon.$ We prove this by contradiction, that is, assume it is not, then for $\varepsilon > 0$, there exist subsequences $\{m(q)_v\}$ and $\{n(q)_v\}$ such that for any positive integer v, $n(q)_v$ is very small, that is

$d(y_{m(q)v}, y_{n(q)v}) \ge \varepsilon,\ d(y_{m(q)v}, y_{n(q)v-1}) < \varepsilon.$

Hence, $\int_0^{\varepsilon} \varphi(t)dt \le \int_0^{d(y_{m(q)v},y_{n(q)v})} \varphi(t)dt \le k\int_0^{d(y_{m(q)v-1},y_{n(q)v-1})} \varphi(t)dt$

As $v \to \infty$, we obtain $\int_0^{\varepsilon} \varphi(t)dt \le k\int_0^{\varepsilon} \varphi(t)dt,$ which is a contradiction.

Hence $d(y_{m(q)v-1}, y_{n(q)v-1}) \le \varepsilon.$ This shows that $\{y_n\}$ is a Cauchy sequence in $X.$

The sequence $\{y_n\} = \{Sx_{n+1}\} \subset S(X)$ is a Cauchy sequence in $S(X).$

Since $S(X)$ is complete, it converges to a point $p = Su$ for some $u \in X.$

Hence the subsequence Tx_{n+1} also converges to $p = Tu,$ for some $u \in X.$

Using (2.1), we have $\int_0^{d(Tu,Tx_{n+1})} \varphi(t)dt \le k\int_0^{d(Su,Sx_{n+1})} \varphi(t)dt.$

Taking limit as $n \to \infty,$ with $0 \le k < 1,$ we have

$$\int_0^{d(Tu,p)} \varphi(t)dt = 0. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.3)$$

By the condition in theorem 2.1, we have from (2.3), that

$d(Tu, p) = 0$ or $Tu = p,$ hence $Tu = Su = p.$

Since S and T are weakly compatible, we have

$STu = TSu,$ that is $Sp = Tp.$

If $Tp \ne p,$ using (2.1), we have

$\int_0^{d(Tp,p)} \varphi(t)dt = \int_0^{d(Tp,Tx_{n+1})} \varphi(t)dt \le k\int_0^{d(Sp,Sx_{n+1})} \varphi(t)dt = k\int_0^{d(Sp,p)} \varphi(t)dt$ which is a contradiction.

Hence $Tp = p.$

Therefore $Tp = Sp = p.$

To prove uniqueness, suppose $z \ne p,$ is also a common fixed point of T and S, then

$$\int_0^{d(p,z)} \varphi(t)dt = \int_0^{d(Tp,Tz)} \varphi(t)dt \le k\int_0^{d(Sp,Sz)} \varphi(t)dt = k\int_0^{d(p,z)} \varphi(t)dt$$

That is $(1-k)\int_0^{d(p,z)} \varphi(t)dt = 0$

$$\Rightarrow \int_0^{d(p,z)} \varphi(t)dt = 0.$$

By the condition in theorem (2.1), we have $d(p,z) = 0.$

Therefore $p = z,$ and the common fixed point is unique. This ends the proof.

Theorem 2.1 leads to the following corollary, if $Sx = x$ and $Sy = y$ (i.e S is an identity mapping)**:**

Corollary 2.2 (Branciari [1]): Let (X,d) be a complete metric space, $c \in (0,1)$ and $f : X \to X$ be a mapping such that for each $x, y \in X,$

$\int_0^{d(fx,fy)} \varphi(t)dt \le c\int_0^{d(x,y)} \varphi(t)dt$, where $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable (i.e with finite integral) on each compact subset of $[0,+\infty),$ nonnegative , and such that for each $\varepsilon > 0,$ $\int_0^{\varepsilon} \varphi(t)dt > 0,$ then f has a unique fixed point $p \in X$, such that for each $x \in X,$ $\lim_{n\to\infty} f^n x = p.$

Theorem 2.3: Let (X,d) be a complete metric space, $c \in (0,1)$ and $f : X \to X$ be a mapping such that for each $x, y \in X,$ $\int_0^{d(fx,fy)} \varphi(t)dt \le c\int_0^{d(x,y)} \varphi(t)dt$, (2.4)

where $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable (i.e with finite integral) on each compact subset of $[0,+\infty),$ nonnegative , and such that for each $\varepsilon > 0,$ $\int_0^{\varepsilon} \varphi(t)dt > 0,$ then f has a unique fixed point $p \in X$, such that for each $x \in X,$ $\lim_{n\to\infty} f^n x = p.$

Proof:

Let $x_0 \in X$, choose $x_1 \in X$ and define $Sx_1 = Tx_0$ since $T(X) \subseteq S(X).$

We define a sequence $\{y_n\}$ in X such that

$$y_n = Sx_{n+1} = Tx_n, \text{ for } n = 0,1,2,...$$

From 2.4, for each $n \geq 1$, we have

$$\int_0^{d(y_n,y_{n+1})} \varphi(t)dt = \int_0^{d(Tx_n,Tx_{n+1})} \varphi(t)dt \leq k\int_0^{e(Sx_n,Sx_{n+1})} \varphi(t)dt, \quad \text{.............................} \quad (2.5)$$

where $e(Sx_n, Sx_{n+1}) = \max\{d(Sx_n, Sx_{n+1}), d(Sx_n, Tx_n), d(Sx_{n+1}, Tx_{n+1}), \frac{[d(Sx_n,Tx_{n+1})+d(Sx_{n+1},Tx_n)]}{2}\}$

that is $e(y_{n-1}, y_n) = \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{[d(y_{n-1},y_{n+1})+d(y_n,y_n)]}{2}\}.$ ....(2.6)

Note that $\frac{d(y_{n-1},y_n)}{2} \leq \frac{[d(y_{n-1},y_n),d(y_n,y_{n+1})]}{2}.$ ....(2.7)

Hence $e(y_{n-1}, y_n) = \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}.$ ....(2.8)

Substituting (2.8) into (2.5), we have

$$\int_0^{d(y_n,y_{n+1})} \varphi(t)dt \leq \int_0^{\max\{d(y_{n-1},y_n)+d(y_n,y_{n+1})\}} \varphi(t)dt,$$

$$\leq k\int_0^{d(y_{n-1},y_n)} \varphi(t)dt,$$

$$\leq k^2\int_0^{d(y_{n-2},y_{n-1})} \varphi(t)dt,$$

$$\leq k^3\int_0^{d(y_{n-3},y_{n-2})} \varphi(t)dt,$$

$$\leq k^n\int_0^{d(y_0,y_1)} \varphi(t)dt. \quad \text{................ ....} \quad (2.9)$$

Taking the limit of (2.9) as $n \to \infty$, with $0 \leq k < 1$, we have

$$\lim_{n\to\infty}\int_0^{d(y_n,y_{n+1})} \varphi(t)dt = 0. \quad \text{...} (2.10)$$

By the condition in theorem 2.3, we have $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$

Next, we show that $\{y_n\}$ is a Cauchy sequence in X.

The sequence $\{y_n\} = \{Sx_{n+1}\} \subset S(X)$ is a Cauchy sequence in $S(X)$.

Since $S(X)$ is complete, it converges to a point $p = Su$ for some $u \in X$. Hence the subsequence Tx_{n+1} also converges to $p = Tu$, for some $u \in X$.

In view of (2.4) and (2.10), we have

$$\int_0^{d(Tu,Tx_{n+1})} \varphi(t)dt \leq k\int_0^{d(Su,Sx_{n+1})} \varphi(t)dt. \quad \text{...} (2.11)$$

Taking limit of (2.11) as $n \to \infty$, with $0 \leq k < 1$, we have

$$\int_0^{d(Tu,p)} \varphi(t)dt = 0. \quad \text{...} (2.12)$$

By the condition in theorem 2.3, we have from (2.12), that

$d(Tu, p) = 0$, hence $Tu = Su = p$.

Since S and T are weakly compatible, we have

$STu = TSu$, that is $Sp = Tp$.

If $Tp \neq p$, then we have

$$\int_0^{d(Tp,p)} \varphi(t)dt = \int_0^{d(Tp,Tx_{n+1})} \varphi(t)dt \leq k\int_0^{d(Sp,Sx_{n+1})} \varphi(t)dt = k\int_0^{d(Sp,p)} \varphi(t)dt$$

But $Sp = Tp,$

Hence $\int_0^{d(Tp,p)} \varphi(t)dt = k\int_0^{d(Tp,p)} \varphi(t)dt$ which is a contradiction,

Thus $Tp = p,$ therefore $Tp = Sp = p.$ ……………………………………(2.13)

To prove uniqueness, suppose $z \neq p,$ is also a common fixed point of T and S, then

$$\int_0^{d(p,z)} \varphi(t)dt = k\int_0^{d(Tp,Tz)} \varphi(t)dt$$

$$\leq k\int_0^{d(Sp,Sz)} \varphi(t)dt = k\int_0^{d(p,z)} \varphi(t)dt. \quad \text{…………………………..(2.14)}$$

Then $p = z$ and the common fixed point is unique.

If we replace equation (2.4) with the following contraction map

$m(Sx,Sy) = \max\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), d(Sx,Ty), d(Sy,Tx)\}$, for all $x, y \in X$, where $0 \leq k < 1.$

Let $y_n = Sx_{n+1} = Tx_n$, then

$$\int_0^{d(y_n,y_{n+1})} \varphi(t)dt \leq k\int_0^{m(y_{n-1},y_n)} \varphi(t)dt, \text{ where} \quad \text{………………………………..(2.15)}$$

$$m(y_{n-1}, y_n) = \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1}), d(y_{n-1}, y_{n+1}), d(y_n, y_n)\}.$$

By triangular inequality,

$$d(y_{n-1}, y_n) = \{d(y_{n-1}, y_n) + d(y_n, y_{n+1})\}. \quad \text{…………………………………….(2.16)}$$

Substituting (2.15) into (2.16), we have

$$\int_0^{d(y_n,y_{n+1})} \varphi(t)dt \leq k\int_0^{[d(y_{n-1},y_n)+d(y_n,y_{n+1})]} \varphi(t)dt.$$

$$\leq k\int_0^{d(y_{n-1},y_n)} \varphi(t)dt. + k\int_0^{d(y_n,y_{n=1})} \varphi(t)dt.$$

$$\Rightarrow \quad (1-k)\int_0^{d(y_n,y_{n+1})} \varphi(t)dt \leq k\int_0^{d(y_{n-1},y_n)} \varphi(t)dt.$$

or $\int_0^{d(y_n,y_{n+1})} \varphi(t)dt \leq \frac{k}{1-k}\int_0^{d(y_{n-1},y_n)} \varphi(t)dt.$

Let $c = \dfrac{k}{1-k}$, then we have

$$\int_0^{d(y_n,y_{n+1})} \varphi(t)dt \leq c\int_0^{d(y_{n-1},y_n)} \varphi(t)dt.$$

$$\int_0^{d(Tu,Tx_{n+1})} \varphi(t)dt \leq c\int_0^{d(Su,Sx_{n+1})} \varphi(t)dt. \text{……………………………………………(2.17)}$$

Taking limit $n \to \infty$, with $c \in [0, \frac{1}{2})$, then

$$\int_0^{d(Tu,p)} \varphi(t)dt = 0.$$

Hence by the condition in theorem 2.3, we have

$$d(Tu, p) = 0.$$

Therefore $Tu = Su = p.$

If $Tp \neq p,$ using (2.17), we have

$$\int_0^{d(Tp,p)} \varphi(t)dt = \int_0^{d(Tp,Tx_{n+1})} \varphi(t)dt \leq c\int_0^{d(Sp,Sx_{n+!})} \varphi(t)dt = c\int_0^{d(Sp,p)} \varphi(t)dt.$$

But $Sp = Tp$, hence $\int_0^{d(Tp,p)} \varphi(t)dt = \int_0^{d(Tp,p)} \varphi(t)dt.$

This is a contradiction, hence $Tp = p.$

Therefore $Tp = Sp = p.$

The uniqueness follows from equation (2.13) and (2.14). This ends the proof.
Theorem 2.3 leads to the following corollary if $Sx = x$ and $Sy = y$ (i.e S is an identity mapping):

Corollary 2.4 (Rhoades [2]): Let (X, d) be a metric space, $k \in [0,1)$ and $T : X \to X$ a self mapping of X such that for all $x, y \in X$, satisfying

$$\int_0^{d(Tx,Ty)} \varphi(t)d(t) \le k \int_0^{m(x,y)} \varphi(t)d(t),$$

where $m(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{[d(x,Ty)+d(y,Tx)]}{2}\}$, where $\varphi : R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, nonnegative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)d(t) > 0.$ Then T has a unique fixed point p in X such that, for each $x \in X$, $\lim_{n\to\infty} T^n x = p.$

Remark 2.5: (i). If $\varphi(t)dt = 1$, in (2.1), we have $d(Tx,Ty) \le kd(Sx,Sy)$, for $0 \le k < 1$, which is the Jungck contraction map.

(ii). If $\varphi(t)dt = 1$, in (2.4), we have $\int_0^{d(Tx,Ty)} \varphi(t)dt = d(Tx,Ty) \le k\int_0^{e(Sx,Sy)} \varphi(t)dt = ke(Sx,Sy)$, for all $x, y \in X$ and $k \in [0,1)$.

(iii). If $\varphi(t)dt = 1$ in (2.4), we have

$\int_0^{d(Tx,Ty)} \varphi(t)dt = d(Tx,Ty) \le k\int_0^{m(Sx,Sy)} \varphi(t)dt = km(Sx,Sy)$, for all $x, y \in X$ and $k \in [0,1)$.

Therefore every contractive condition of integral type also include a corresponding contractive condition not involving integrals by setting $\varphi(t) = 1$ over R^+.

Conclusion

Theorem 2.1 has significantly improved the Jungck's fixed point theorem by employing weakly compatible maps instead of commutativity of maps. This research paper has also extended and generalized the theorem of Branciari [1] and Rhoades [2] to a pair of weakly compatible mappings in a metric space.

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