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Stability of Triangular Points of the Photogravitational Robe's Restricted Three-Body Problem

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Abstract

The linear stability of triangular points was studied for the Robe's restricted three-body problem when the bigger primary (rigid shell) and the second primary are radiating. The critical mass obtained depends on the radiation of both the first (rigid shell) and the second primaries respectively, as well as the density parameter k. The stability of the triangular points depends largely on the values of k. The destabilizing tendency of the radiation factors were enhanced when k > 0 and weakened for k < 0.

Keywords: stability, Triangular points, Robe's problem, Density parameter.

1.0 Introduction

Robe [1] introduced a new kind of restricted three-body problem that incorporates the effect of buoyancy force. One of the primaries m_1 is a rigid shell of mass filled with homogeneous incompressible fluid of density ρ_1. The second primary m_2 is a point mass located outside the shell. The third body m_3 is the particle of negligible mass of density ρ_3 which moves inside the shell under the influences of the gravitational attraction of the primaries and the buoyancy force of the fluid of density ρ_1. Robe studied the motion of the infinitesimal mass when m_2 describes both circular and elliptic orbits. He obtained the equilibrium points and showed that, for the circular case, the equilibrium point is linearly stable when $\rho_3 < \rho_1$ and unstable when $\rho_1 < \rho_3$.

Robe's model may be useful for studying the small oscillations of the earth's inner core by taking into consideration the moon's attraction. The model is also applicable to the study of the motion of artificial satellites under the influence of earth's attraction.

Robe's problem has been modified to define a new problem [2 – 6].

In this model we consider both the first (rigid shell) and the second primaries radiating, to study the effect of radiation on the stability of the triangular equilibrium points of the Robe's restricted three-body problem.

The paper consists of four sections. Section one establishes the relevant equations of motion that incorporates the effect of buoyancy force using some basic assumptions. In the second section we obtained the equilibrium points. Section three deals with the variational equations of motion of the problem and solutions of the resulting characteristic equation obtained. In section four, we obtained the critical mass of the mass parameter. This is followed by the conclusion on the findings.

2.0 Equations of Motion

Let the mass of the rigid shell be m_1 and the point mass be m_2. Let the density of the incompressible fluid inside the shell be ρ_1 and that of the infinitesimal mass be ρ_3 and it's mass m. Let q_1 , q_2 denote the radiation coefficients of the first and second primaries, respectively, which are given

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by $F_{\rho_i} = F_g\,(1-q_i)$ such that $0 < 1-q_i << 1,\ (i = 1, 2)$.

Let M_1, M_2 and M_3 be the centres of m_1, m_2 and m respectively such that $M_1 M_3 = r_1$ and $m_2\ m_3 = r_2$. Let G be the gravitational constant and (x,y) the coordinates of the infinitesimal mass m. let the line joining m_1 *and* m_2 be the x – axis. Then the total potential acting on m is

$$-\frac{Gm_2 q_2}{r_2} + \frac{4}{3}\pi G\,\rho_1\left(1-\frac{\rho_1}{\rho_3}\right)r_1^2 - \frac{Gm_1 q_1}{r_1} \tag{1}$$

where

$$\begin{aligned} r_1^2 &= (x-x_1)^2 + y^2 \\ r_2^2 &= (x-x_2)^2 + y^2 \end{aligned} \tag{2}$$

Let the coordinates of m_1 and m_2 be $(x_1, 0)$ and $(x_2, 0)$ respectively. In the dimensionless rotational coordinates system we choose the unit of mass to be sum of the masses of the primaries $(m_1 = 1-\mu \text{ and } m_2 = \mu)$. We take the unit of length equal to the distance between the primaries and is chosen such that $G = 1$.

The equations of motion of the infinitesimal body are [7]

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x \\ \ddot{y} + 2\dot{x} &= \Omega_y \end{aligned} \tag{3}$$

where

$$\Omega = \frac{1}{2}\left(x^2+y^2\right) - kr_1^2 + \frac{1-\mu}{r_1}q_1 + \frac{\mu}{r_2}q_2 \tag{4}$$

$$k = \frac{4}{3}\pi\rho_1\left(1-\frac{\rho_1}{\rho_3}\right),\ \mu = \frac{m_2}{m_1+m_2} : 0 < \mu \le \frac{1}{2},$$

$$\begin{aligned} r_1^2 &= (x+\mu)^2 + y^2 \\ r_2^2 &= (x-1+\mu)^2 + y^2 \end{aligned} \tag{5}$$

3.0 Equilibrium Points

Equilibrium points exist when

$$\Omega_x = \Omega_y = 0 \tag{6}$$

For $k \neq 0$ we have

$$\Omega_x = x - 2k(x+\mu) - \frac{(1-\mu)(x+\mu)}{r_1^3}q_1 - \frac{\mu(x-1+\mu)}{r_2^3}q_2 \tag{7}$$

$$\Omega_y = y - 2ky - \frac{(1-\mu)y}{r_1^3}q_1 - \frac{\mu y}{r_2^3}q_2\ . \tag{8}$$

3.1 Triangular Points

The triangular points are given by the equations

$$\Omega_x = 0,\ \Omega_y = 0,\ y \neq 0.$$

That is

$$x\left[1-2k-\frac{1-\mu}{r_1^3}q_1 - \frac{\mu}{r_2^3}q_2\right] - \mu\left[2k+\frac{1-\mu}{r_1^3}q_1 - \frac{1-\mu}{r_2^3}q_2\right] = 0 \tag{9}$$

and

$$y\left[1-2k-\frac{1-\mu}{r_1^3}q_1 - \frac{\mu}{r_2^3}q_2\right] = 0. \tag{10}$$

Equations (9) and (10) give

$$1-2k-\frac{q_1}{r_1^3}=0 \tag{11}$$

$$1-2k-\frac{q_2}{r_2^3}=0 \tag{12}$$

Knowing r_1 and r_2 from equations (11) and (12) the exact coordinates of the triangular points are obtained by using equations (5) for x and y.
Thus

$$x=\frac{1}{2}-\mu+\frac{1}{2}\left(r_1^2-r_2^2\right)$$

$$y=\pm\left[\frac{1}{2}\left(r_1^2+r_2^2\right)-\frac{1}{4}-\frac{1}{4}\left(r_1^2-r_2^2\right)^2\right]^{\frac{1}{2}} \tag{13}$$

When the bigger primary is not oblate, the smaller primary is not radiating and $\rho_1=\rho_3$

$$r_i=1 \qquad (i=1,2). \tag{14}$$

We assume the solutions of equations (11) and (12) are

$$r_i=1+\alpha_i \qquad (i=1,2) \tag{15}$$

where α_1 *and* α_2 are very small perturbations. Using equations (11) and (12) and restricting ourselves to linear terms in *A* , *1 – q* and *k*, we obtain

$$x=\frac{1}{2}-\mu-\frac{1}{3}\left(\frac{3+4k}{3+2k}\right)\left[q_2-q_1\right]$$

$$y=\pm\sqrt{\frac{9+16k}{12}}\left[1-\frac{1}{2(3+2k)}\left\{2-\left(q_1+q_2\right)\right\}\right]. \tag{16}$$

The coordinates $(x, \pm\, y)$ obtained in equation (16) are the triangular points and are denoted by L_4 and $L_{5.}$

4.0 Stability of Triangular Points

Putting $x=x_0+\xi$, $y=y_0+\eta$ in equations (3), in order to study the motion near the triangular points L_4 and L_5, we obtain the variational equations of motion as

$$\ddot{\xi}-2n\dot{\eta}=\Omega_{xx}\left(x_0, y_0\right)\xi+\Omega_{xy}\left(x_0, y_0\right)\eta \tag{17}$$

$$\ddot{\eta}+2n\dot{\xi}=\Omega_{xy}\left(x_0, y_0\right)\xi+\Omega_{yy}\left(x_0, y_0\right)\eta$$

The characteristics equation is

$$\lambda^4-\left(\Omega_{xx}^o+\Omega_{yy}^o-4n^2\right)\lambda^2+\Omega_{xx}^o\Omega_{yy}^o-\left(\Omega_{xy}^o\right)^2=0 \tag{18}$$

where the superscript ° indicates that the partial derivatives are evaluated at the triangular points $\left(x_0\ ,\ y_0\right)$, and are given by

$$\Omega_{xx}^0=3\left(\frac{1-2k}{3+4k}\right)\left[\frac{3}{4}+a_1+\mu b_1\right]$$

$$\Omega_{yy}^0=3\left(\frac{1-2k}{3+4k}\right)\left[\frac{3}{4}\left(3+\frac{16}{3}k\right)+a_2+\mu b_2\right]$$

$$\Omega_{xy}^0=3\left(\frac{1-2k}{3+4k}\right)\sqrt{3+\frac{16}{3}k}\left[-\frac{3}{4}+a_3+\mu\left(\frac{3}{2}+b_3\right)\right]$$

where

$$a_1 = \frac{1}{4}\left[2\left(\frac{-3-8k}{3+2k}\right)(1-q_1)+4\left(\frac{3+4k}{3+2k}\right)(1-q_2)\right]$$

$$b_1 = \frac{1}{4}\left[2\left(\frac{+9+16k}{3+2k}\right)(1-q_1)-2\left(\frac{9+16k}{3+2k}\right)(1-q_2)\right]$$

$$a_2 = \frac{1}{12}\left(3+\frac{16}{3}k\right)\left[3\left(\frac{2-16k}{3+2k}\right)(1-q_1)-4\left(\frac{3+4k}{3+2k}\right)(1-q_2)\right]$$

$$b_2 = \frac{1}{12}\left(3+\frac{+16}{3}k\right)\left[-\left(\frac{18}{3+2k}\right)(1-q_1)+\left(\frac{18}{3+2k}\right)(1-q_2)\right]$$

$$a_3 = \frac{1}{12}\left[6\left(\frac{1+4k}{3+2k}\right)(1-q_1)-4\left(\frac{3+4k}{3+2k}\right)(1-q_2)\right]$$

$$b_3 = \frac{1}{4}\left[2\left(\frac{3-4k}{3+2k}\right)(1-q_1)+2\left(\frac{3-4k}{3+2k}\right)(1-q_2)\right] \qquad (19)$$

Each $|a_i|, |b_i|$ is very small.

The characteristic equation becomes

$$\lambda^4-\left[\left\{3\left(\frac{1-2k}{3+4k}\right)(b_1+b_2)\right\}\mu+9\left(\frac{1-2k}{3+4k}\right)-4+3\left(\frac{1-2k}{3+4k}\right)\left(\frac{16}{3}k+a_1+a_2\right)-6A\varphi^2\right]\lambda^2$$

$$-\frac{9}{4}\left(\frac{1-2k}{3+4k}\right)^2\left(3+\frac{16}{3}k\right)\left[3(3+4b_3)\mu^2+\left(-9-3b_1-\frac{3b_2}{3+\frac{16}{3}k}+12a_3-6b_3\right)\mu\right]G=0$$

where

$$G=\left(-3a_1-\frac{3b_2}{3+\frac{16}{3}k}+6a_3\right)$$

The Roots of the Characteristics equation are

$$\lambda^4-\left[3\left(\frac{1-2k}{3+4k}\right)(b_1+b_2).\,\mu+9\left(\frac{1-2k}{3+4k}\right)-4+16k\left(\frac{1-2k}{3+4k}\right)+3\left(\frac{1-2k}{3+4k}\right)a_1+3\left(\frac{1-2k}{3+4k}\right)a_2\pm\sqrt{\Delta}\right]$$

We observe that the roots are functions of μ, q_1, q_2, k and they depend upon the nature of the discriminant Δ and is given by

$$\Delta=9\left(1-\frac{20}{3}k\right)\left(1+\frac{16}{9}k\right)(3+4b_3)\,\mu^2$$

$$+\left[\ \left\{-2\left(1-\frac{10}{3}k\right)\left(1+\frac{14}{3}k\right)-9\left(1-\frac{20}{3}k\right)\left(1+\frac{19}{9}k\right)\right\}b_1\right.$$

$$\left\{-2\left(1-\frac{10}{3}k\right)\left(1+\frac{14}{3}k\right)-3\left(1-\frac{28}{3}k\right)\right\}b_2+18\left(1-\frac{20}{3}k\right)\left(1+\frac{16}{9}k\right)b_3$$

$$+36\left(1-\frac{20}{3}k\right)\left(1+\frac{16}{9}k\right)a_3-27\left(1-\frac{20}{3}k\right)\left(1+\frac{16}{9}k\right)\ ]\mu$$
$$+\left(1+\frac{14}{3}k\right)^2+\left\{-2\left(1-\frac{10}{3}k\right)\left(1+\frac{14}{3}k\right)-9\left(1-\frac{20}{3}k\right)\left(1+\frac{16}{9}k\right)\right\}a_1$$
$$+\left\{-2\left(1-\frac{10}{3}k\right)\left(1+\frac{14}{3}k\right)-3\left(1-\frac{28}{3}k\right)\right\}a_2-18\left(1-\frac{20}{3}k\right)\left(1+\frac{16}{9}k\right)a_3 \tag{20}$$

Three cases can be discussed for Δ:

1. When $\Delta > 0$, we have that the roots are negative showing that the triangular points are linearly stable.
2. When $\Delta < 0$, we have that the real parts of two of the four roots are positive and equal, showing that the triangular points are unstable.
3. When $\Delta = 0$, we have that the double roots give secular terms, showing that the triangular points are unstable.

5.0 Critical Mass

The solution of the equation $\Delta = 0$ gives the critical mass value μ_c of the mass parameter. That is

$$\mu_c=\frac{1}{2}\left(1-\sqrt{\frac{P-4Q}{P}}\right)$$
$$\frac{1}{2}\left[-2\left(1-\frac{10}{3}k\right)\left(1+\frac{14}{3}k\right)-9\left(1-\frac{20}{3}k\right)\left(1+\frac{19}{9}k\right)\right]\left[\frac{2a_1+b_1}{\sqrt{P(P-4Q)}}-\frac{b_1}{P}\right]$$
$$\frac{1}{2}\left[-2\left(1-\frac{10}{3}k\right)\left(1+\frac{14}{3}k\right)-9\left(1-\frac{20}{3}k\right)\left(1+\frac{19}{9}k\right)\right]\left[\frac{2a_2+b_2}{\sqrt{P(P-4Q)}}-\frac{b_2}{P}\right]$$
$$\frac{1}{3}\left[\frac{4Q}{\sqrt{P(P-4Q)}}+2\,\frac{\sqrt{P-4Q}}{\sqrt{P}}-\frac{\sqrt{P}}{\sqrt{P(P-4Q)}}-1\right]b_3-\frac{2}{3}a_3 \tag{21}$$

where $P=9\left(1-\frac{20}{3}k\right)\left(3+\frac{16}{3}k\right)$ *and* $Q=\left(-1-\frac{14}{3}k\right)^2$. Restricting ourselves to linear terms in $k, 1-q_1$ *and* $1-q_2$, we find that

$$\mu_c=\mu_0+\mu_r+\mu_p \tag{22}$$

where

$$\mu_0=\frac{1}{2}\left(1-\sqrt{\frac{23}{27}}\right)$$
$$\mu_r=-\frac{4}{27\sqrt{69}}+\frac{2}{27\sqrt{69}}\left(q_1+q_2\right)$$
$$\mu_p=-\frac{128}{27\sqrt{69}}k+\frac{100}{27}\left(1-\frac{1043771}{27600\sqrt{69}}\right)k\left(1-q_1\right)-\frac{64}{27}\left(1+\frac{225073}{2208\sqrt{69}}\right)k\left(1-q_2\right)$$

Equation (22) gives the critical mass value μ_c of the mass parameter. It reflects the effect of the radiation of the first primary (rigid mass) and the second primary on the critical mass of the Robe's restricted three-body problem, indicating a destabilizing effect on the triangular equilibrium points.

The destabilizing tendency of the radiating factor is further enhanced when k > 0 $(\rho_1 < \rho_3)$ and weakened when k > 0 $(\rho_1 > \rho_3)$.

When k = 0 we confirm the result of AbdulRaheem and Singh [7] for $\varepsilon_1=\varepsilon_2=0,\ A_1=0$ *and* $A_2=0$.

When $k=0,\ q_1=1=q_2$ we obtain the critical mass value of the classical restricted three-body problem.

6.0 Conclusion

The effect of radiation in the first (rigid shell) and second primaries on the stability of the triangular equilibrium points of the Robe's restricted three-body problem has been studied. The value of the critical mass value obtained depends on the radiation factors of the rigid shell and that of the second primary respectively, and the density of the fluid and that of the infinitesimal mass in the shell.

It was observed that the radiation factors have destabilizing tendencies on the triangular equilibrium points. These destabilizing tendencies are further enhanced or weakened, depending on whether the density of the fluid in the shell is less than that of the infinitesimal mss or the density of the infinitesimal mass is less than that of the fluid.

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