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Mathematical Analysis of the Dynamics of a 3 mass-spring system with connecting dashpot topology having unequal coupled masses

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Abstract

The Variational Iteration approach is used to analyzing nonlinear forced vibration of a nonconservative three mass-spring system having mass-mass dashpot. By implementing suitable intermediate variables, three nonlinear differential (Duffing) equations of the system are transformed into a single nonlinear differential equation. Hence, the Method is used to find the difference between the two extreme displacement, of which the result is used to find the individual displacements of the three masses by using the analytical method. The initial value of difference between the two extreme displacement is obtained by eliminating secular term 'tsint' and 'tcost', hence the three masses are varied to check its effect on the system and to investigated how the size of one mass affect the dynamics of its of other masses.

Keywords: Nonlinear Forced Vibration, Variational Iteration Method; Three-degree-freedom, Analytical relation.

1.0 Introduction

The studies conducted before, show to analyze nonlinear oscillation system, its needed to understand the motions of nonlinear single-and two-degree-of-freedom oscillation systems, deeply. The three-degree-of-freedom (TDOF) oscillation system are mainly modeled with three coupled non-homogeneous ordinary differrential equation. Nonlinear oscillator with more than one degree-of-fredom are considerably more complicated than those with only one degree-of-freedom.

The nonlinear free vibration of a two-mass system with two degrees of freedom is discussed by Cveticanin[1] and for the case when the non-linearity is of a cubic type, and the analytical solution of the system is obtained. An analytical approach is developed for the nonlinear oscillation of a conservative, Two Degrees of freedom mass-spring system with serial combined linear and nonlinear stiffness excited by a constant external force by Lim et al [2]. Lai and Lim[3] applied an analytical approach for nonlinear free vibration of a conservative system, suitable intermediate variables which transform two nonlinear differential equations of a two-mass system into a nonlinear differential equation. Razavi[4] applied multiple scales method to solve non-linear forced vibration of non conservative two degree of freedom mass-spring system having linear and nonlinear stiffness.

Kawamura et al[5] proposed an analytical approach for nonlinear forced vibration of a multi-degree-of-freedom using the mode synthesis method.

In this paper, the nonlinear forced vibration of a nonconversative Three Degree of freedom mass-spring system having linear and nonlinear stiffness is studied and the effect of the masses the system is analyzed.

2.0 Mathematical formulation of problem

From fig.1, c is damping coefficient, k_1 and k_2 are linear stiffness while k_3 is the nonlinear stiffness, Ω is excitation frequency, m_1, m_2, m_3 are masses while f_0 is excitation amplitude.

An Ideal Mass-Spring-Damper system with mass, spring constant (linear and nonlinear stiffness) and viscous damping of damping coefficient is subjected to the following;

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$$U_S = -kd \tag{1}$$

Spring force

$$U_D = -c(\dot{l} - \dot{d}) \tag{2}$$

Damping force

$$U_v = m\ddot{d} \tag{3}$$

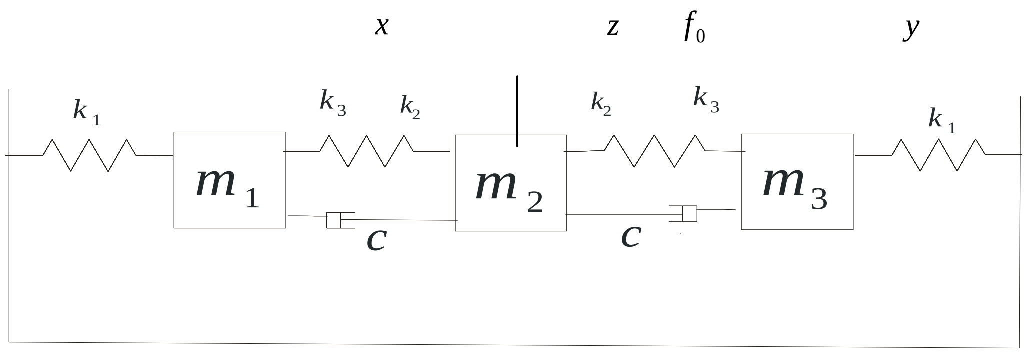

Fig 1 Non Conservative Mass-Spring system with mass-mass Dashpot

Propelling force

$$U_T = k_a(d-l) \tag{4}$$

Opposing Spring force

$$U_N = k_b((d-l)^3 + 3(d-l)^2.(l-h) \tag{5}$$

Note: Nonlinear force of a coupled mass-spring-damper is given by,

$$U_N = k_b((d-l)^3 + \frac{\partial}{\partial(d-l)}(d-l)^3..(l-h) \tag{6}$$

Where $l-h$ is opposite distance due to moment of the force.

Using the above for mass m_1,

$$U_v = U_S + U_D + U_N + U_T \rightarrow U_v - U_S - U_D - U_N - U_T = 0 \tag{7}$$

$$\ddot{u} + c(\dot{w} - \dot{u}) + k_1 u + k_2(w-u) + k_3\left[(w-u)^3 + 3(w-u)^2(u-w)\right] = 0 \tag{8}$$

Also, for mass m_2

$$U_N = 0 \therefore \ddot{w} + cw + k_2(w+u) + cw + k_2(w-v) = 0 \tag{9}$$

$$\ddot{w} + 2cw + 2k_2 w = k_2(v - w + w - u) = k_2(v-u) \tag{10}$$

Also, for mass m_3,

$$U_v - U_S - U_D - U_N - U_T = f_0 \cos\Omega t \tag{11}$$

$$\ddot{v} + c(\dot{w} - \dot{v}) + k_1 v + k_2(w-v) + k_3\left[(w-v)^3 + 3(w-v)^2(v-w)\right] = f_0 \cos\Omega t \tag{12}$$

$$\eta_1 = \frac{c}{m_1}, \eta_2 = \frac{k_1}{m_1}, \eta_3 = \frac{k_2}{m_1}, \eta_4 = \frac{k_3}{m_1}, F = \frac{f_0}{m_2}, \eta_5 = \frac{c}{m_2},$$

$$\eta_6 = \frac{k_1}{m_2}, \eta_7 = \frac{k_2}{m_2}, \eta_8 = \frac{k_3}{m_2}, \eta_9 = \frac{c}{m_3},$$

$$\eta_{10} = \frac{k_1}{m_3}, \eta_{11} = \frac{k_2}{m_3}, \eta_{12} = \frac{k_3}{m_3}$$

Equations (8),(10) and (12) becomes,

$$\ddot{u}+\eta_1(\dot{w}-\dot{u})+\eta_2 u+\eta_3(w-u)+\eta_4\left[(w-u)^3+3(w-u)^2(v-w)\right]=0 \tag{13}$$

$$\ddot{v}+\eta_5(\dot{w}-\dot{v})+\eta_6 v+\eta_7(w-v)+\eta_8\left[(w-v)^3+3(w-v)^2(u-w)\right]=F\cos\Omega t \tag{14}$$

$$\ddot{w}+2\eta_9\dot{w}+2\eta_{11}w=\eta_{11}(v-w+w-u)=\eta_{11}(v-u) \tag{15}$$

3.0 Solution to Duffing equations using Variational Iteration Method

Using initial conditions

$v(0)-u(0)=r_1$ *Initial difference between displacements at extreme points*

$v(0)-w(0)=r_2$

$w(0)-u(0)=r_3$

$\dot{v}(0)=\dot{u}(0)=0$ *Initial velocity*

$v(0)=r_4$ *Initial displacement*

$w(0)=r_2$ *Initial displacement*

$\dot{w}(0)=0$ *Initial velocity*

Using intermediate variables,

$u=x, \quad w=z+u$, and $v=y+w$

where we

let $v-u=p, v-w=q, w-u=s$

Subtract (13) from (14), we have;

$$\ddot{p}-\eta_1\dot{p}+(\eta_2-\eta_3)p-\eta_4 p^3=F\cos\Omega t-(\eta_1-\eta_5)\dot{q}+(\eta_2-\eta_6)v-(\eta_3-\eta_7)q$$
$$-(\eta_4-\eta_8)(q^3+3q^2.s) \tag{16}$$

Let $\eta_2-\eta_3=\sigma^2, \eta_{11}-\eta_7=\delta^2, -\eta_3=\kappa^2, \eta_6-\eta_7=\xi^2$

Using He[6] V.I.M formula for the equation $\ddot{p}+\omega^2 p+f=0$

$$p_{n+1}(t)=p(0)\cos\sigma t+\frac{1}{\sigma}\dot{p}(0)\sin\sigma t+\frac{1}{\sigma}\int_0^t \sin\sigma(s-t).[(\eta_1-\eta_5)\dot{q}_s-\eta_1\dot{p}_s$$
$$+(\eta_3-\eta_7)q_s+(\eta_6-\eta_2)v_s+(\eta_4-\eta_8)(q^3+3q^2.s)]ds \tag{17}$$

Using the initial approximations of the intermediate variables are given as follows;

$$p_0(t)=r_1\cos\sigma t, q_0(t)=r_2\cos\delta t, s_0(t)=r_3\cos\kappa t, v_0(t)=r_4\cos\xi t$$

we have the solution

$$p_{n+1}(t)=r_1\cos\sigma t+\frac{1}{\sigma}[\frac{-\delta(\eta_1-\eta_5)r_2}{(\sigma^2-\delta^2)}(\delta.\sin(\sigma.t)-\sigma.\sin(\delta.t))+\frac{\sigma\eta_1 r_1}{2}(t.\cos(\sigma.t)-\frac{\sin(\sigma.t)}{\sigma})$$
$$\frac{\sigma(\eta_7-\eta_3)r_2}{(\sigma^2-\delta^2)}(\cos(\delta.t)-\cos(\sigma.t))+\frac{\sigma(\eta_2-\eta_6)r_4}{(\sigma^2-\xi^2)}(\cos(\xi.t)-\cos(\sigma.t))+\frac{\sigma(\eta_7-\eta_3)r_2^{\,3}}{4(\sigma^2-9\delta^2)}(\cos(3\delta.t)-\cos(\sigma.t))$$
$$+\frac{3\sigma(\eta_8-\eta_4)r_2^{\,3}}{4(\sigma^2-\delta^2)}(\cos(\delta.t)-\cos(\sigma.t))+\frac{\eta_4 r_1^{\,3}}{8}(\frac{\cos(\sigma.t)-\cos(3\sigma.t)}{4\sigma}-3t.\sin(\sigma.t))$$
$$\frac{3(\eta_8-\eta_4)r_2^{\,2}r_3}{8}[\frac{2(\sigma+\kappa)}{(\sigma+\kappa)^2-(2\delta)^2}(\cos((\kappa+2\delta).t)-\cos(\sigma.t))+\frac{2(\sigma-\kappa)}{(\sigma-\kappa)^2-(2\delta)^2}(\cos((\kappa-2\delta).t)-\cos(\sigma.t))+$$
$$\frac{4\sigma}{\sigma^2-\kappa^2}(\cos(\kappa.t)-\cos(\sigma.t))]+\frac{F}{\sigma^2-\Omega^2}(\cos(\Omega.t)-\cos(\sigma.t))] \tag{18}$$

Eliminating secular terms $t.\sin(\sigma.t)$ and $t.\cos(\sigma.t)$, for σ, η_1, η_4 to exist, $r_1=0 \Rightarrow r_2=-r_3$,

hence this implies that $r_2=r_3=0$

since $v(0)\geq w(0)\geq u(0)$

$$p_{n+1}(t) = (\frac{(\eta_6 - \eta_2)r_4}{(\sigma^2 - \xi^2)} - \frac{F}{\sigma^2 - \Omega^2}).\cos(\sigma.t) + \frac{F}{\sigma^2 - \Omega^2}.\cos(\Omega.t) - \frac{(\eta_6 - \eta_2)r_4}{(\sigma^2 - \xi^2)}.\cos(\xi.t) \quad (19)$$

Substituting the intermediate variable $v - u = p$ in (15)

$$\ddot{w} + 2\eta_1 \dot{w} + 2\eta_3 w = \eta_3(v-u) = \eta_3 p \quad (20)$$

Let $2\eta_3 = \alpha^2$

Using exact method, we have
We have solution of the form

$$w = ae^{n_1 t} + be^{n_2 t} + A\cos\sigma t + B\sin\sigma t + C\cos\Omega t + D\sin\Omega t + E\cos\xi t + F\sin\xi t \quad (21)$$

If $n_1 \neq n_2$

where $n_1 = -\eta_1 + \sqrt{\eta_1{}^2 - \alpha^2}$ and $n_2 = -\eta_1 - \sqrt{\eta_1{}^2 - \alpha^2}$ (22)

$$b = \frac{n_1(A + C + E - r_5) - \sigma B - \Omega D - \xi G}{n_2 - n_1} \quad (23)$$

$$a = \frac{n_2(r_5 - A - C - E) + \sigma B + \Omega D + \xi G}{n_2 - n_1} \quad (24)$$

or

$$w = ae^{n_1 t} + bte^{n_1 t} + A\cos\sigma t + B\sin\sigma t + C\cos\Omega t + D\sin\Omega t + E\cos\xi t + F\sin\xi t \quad (25)$$

where $n_1 = n_2 = -\eta_1$ (26)

$$a = r_5 - A - C - E$$

$$b = n_1(A + C + E - r_5) - \sigma B + \Omega D + \xi G \quad (27)$$

Also,

$$B = \frac{\alpha^2.\sigma.\eta_9}{(\alpha^2 - \sigma^2)^2 + (2\eta_9\sigma)^2}(\frac{(\eta_6 - \eta_2)r_4}{(\sigma^2 - \xi^2)} - \frac{F}{\sigma^2 - \Omega^2}) \quad (28)$$

$$A = \frac{\alpha^2.(\alpha^2 - \sigma^2)}{2((\alpha^2 - \sigma^2)^2 + (2\eta_9\sigma)^2)}(\frac{(\eta_6 - \eta_2)r_4}{(\sigma^2 - \xi^2)} - \frac{F}{\sigma^2 - \Omega^2}) \quad (29)$$

$$D = \frac{F.\alpha^2.\eta_9.\Omega}{(\sigma^2 - \Omega^2).((\alpha^2 - \sigma^2)^2 + (2\eta_9\sigma)^2)} \quad (30)$$

$$C = \frac{F.\alpha^2.(\alpha^2 - \Omega^2)}{2(\sigma^2 - \Omega^2).((\alpha^2 - \sigma^2)^2 + (2\eta_9\sigma)^2)} \quad (31)$$

$$G = \frac{F.\alpha^2.\eta_9.\xi(\eta_2 - \eta_6).r_4}{(\sigma^2 - \xi^2).((\alpha^2 - \xi^2)^2 + (2\eta_9\xi)^2))} \quad (32)$$

$$E = \frac{F(\alpha^2 - \xi^2).\alpha^2.(\eta_2 - \eta_6).r_4}{2(\sigma^2 - \xi^2).((\alpha^2 - \xi^2)^2 + (2\eta_9\xi)^2))} \quad (33)$$

Since w is in between u and v

$$w = \frac{cu + dv}{c + d} \quad (34)$$

and $v - u = p$ (35)

and this yield

$$u = \frac{(c+d)w - dp}{c+d}$$
$$v = \frac{(c+d)w + cp}{c+d} \tag{36}$$

Note: c and d are determined by the initial ratio between the displacement $y\vec{z}$ and $x\vec{z}$

4.0 Numerical application

Equation (18) is observed to consist of the terms *tsint* and *tcost* which are secular terms, hence we deduce that

$$\frac{-\eta_1 r_1}{2} = 0 \tag{37}$$

and

$$-3\frac{\eta_4 r_1^3}{8\sigma} = 0, \tag{38}$$

since η_4, η_1 and σ cannot be zero since damping coefficient, linear and nonlinear stiffness exist, therefore $r_1 = 0$

The initial difference between displacent y and x is zero.

From the above, we observe that for B to exist $\sigma^2 - \Omega^2 \neq 0 \Rightarrow |\sigma| \neq \Omega$

Also, if B exist, for G to exist $\alpha^2 - \xi^2 \neq 0 \Rightarrow |\alpha| \neq \xi$

Also, we take $c = d = 1$

For the analysis of m_1 we use the motion of masses of a typical system Razavi[4] having the following properties: $k_1 = 2.5, k_2 = 0.5, c = 1, \Omega = 1, m_2 = 0.5, m_3 = 1, r_5 = 1, r_4 = 1, f_0 = 1$, varying the value of m_1 we have the following relationship between z, x, y and t.

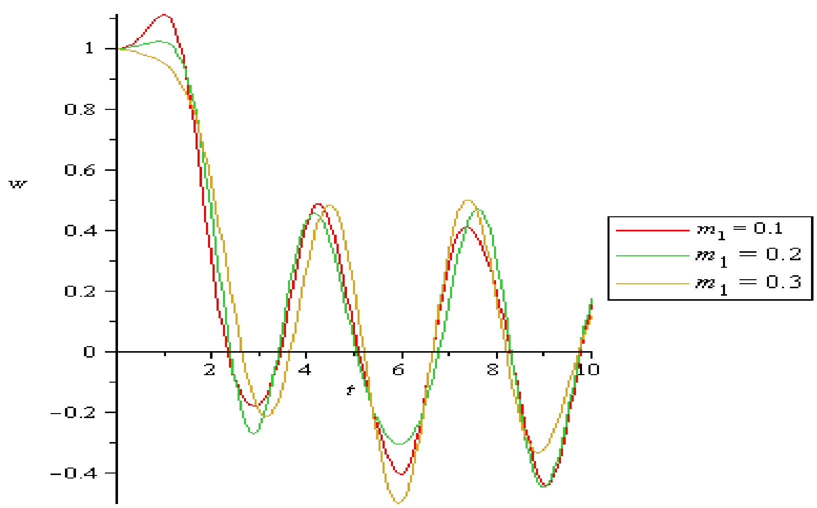

Fig 2 A graph of displacement *w against t*

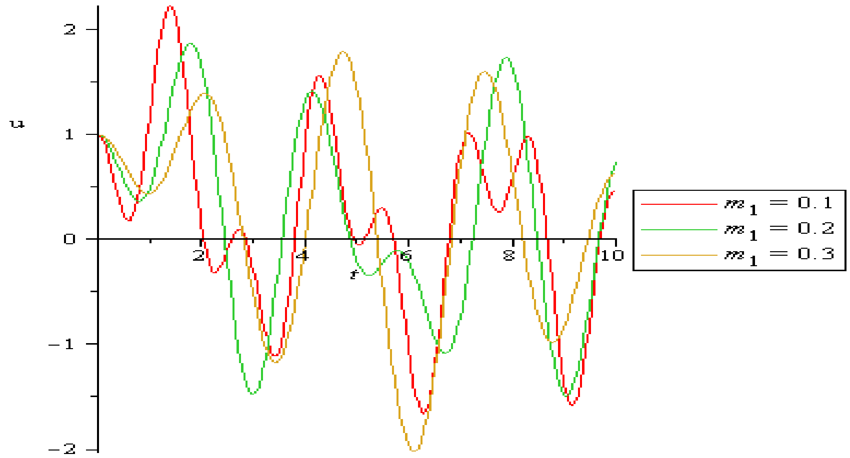

Fig 3 A graph of displacement *u against t*

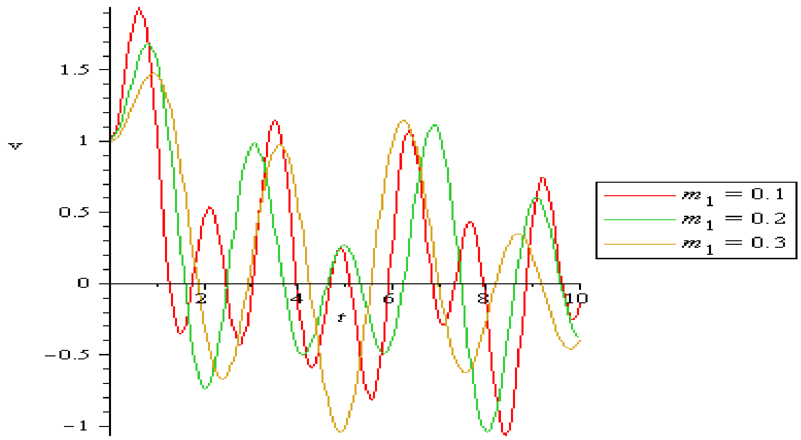

Fig 4 A graph of displacement *v against t*

For the analysis of m_2 we use the motion of masses of a typical system Razavi[4] having the following properties: $k_1 = 2.5, k_2 = 0.5, c = 1, \Omega = 1, m_1 = 0.5, m_3 = 1, r_5 = 1, r_4 = 1, f_0 = 1,$ varying the value of m_1 we have the following relationship between z, x, y and t.

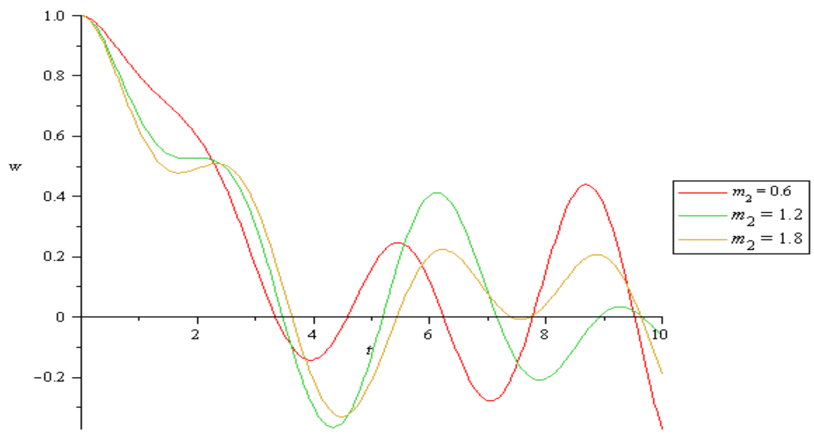

Fig 5 A graph of displacement *w against t*

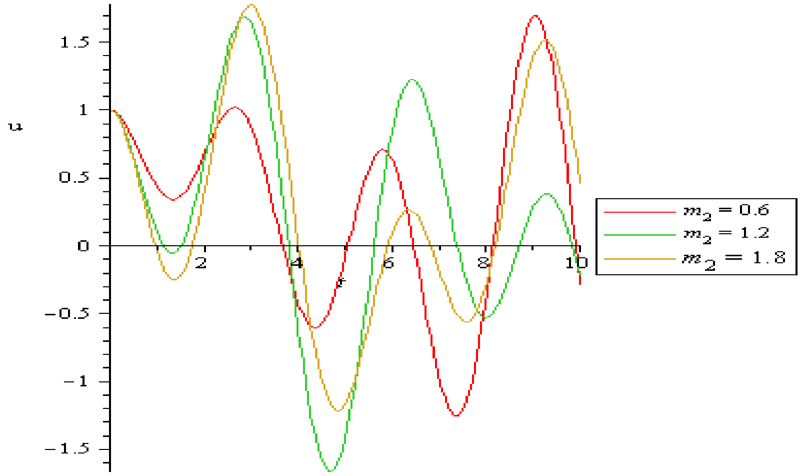

Fig 6 A graph of displacement *u against t*

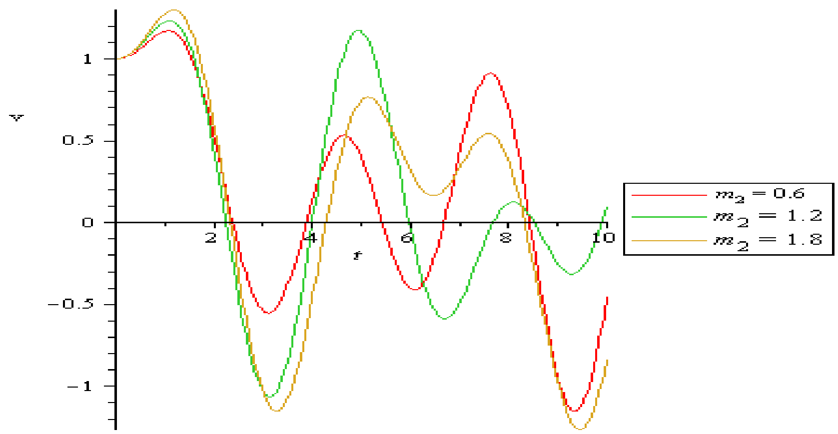

Fig 7 A graph of displacement *v against t*

For the analysis of m_2 we use the motion of masses of a typical system Razavi[4] having the following properties: $k_1 = 2.5, k_2 = 0.5, c = 1, \Omega = 1, m_1 = 0.5, m_2 = 1, r_5 = 1, r_4 = 1, f_0 = 1$, varying the value of m_1 we have the following relationship between z, x, y and t.

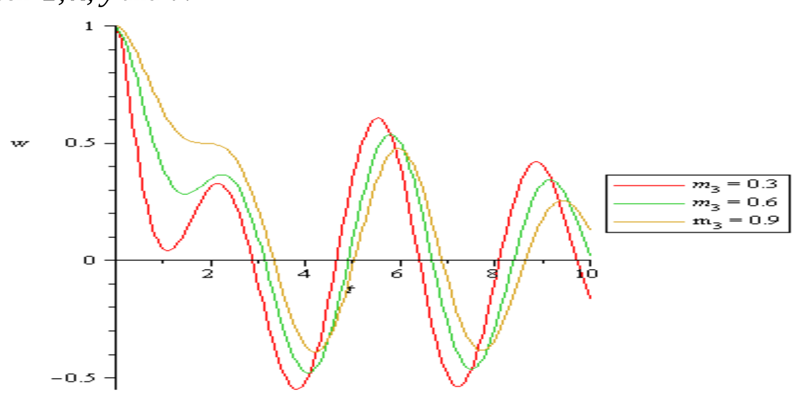

Fig 8 A graph of displacement *w against t*

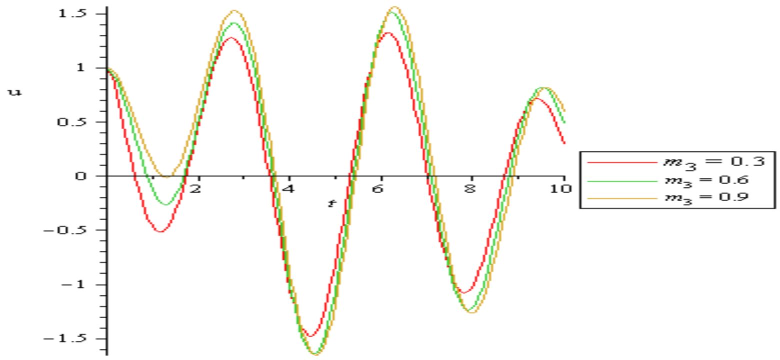

Fig 9 A graph of displacement *u against t*

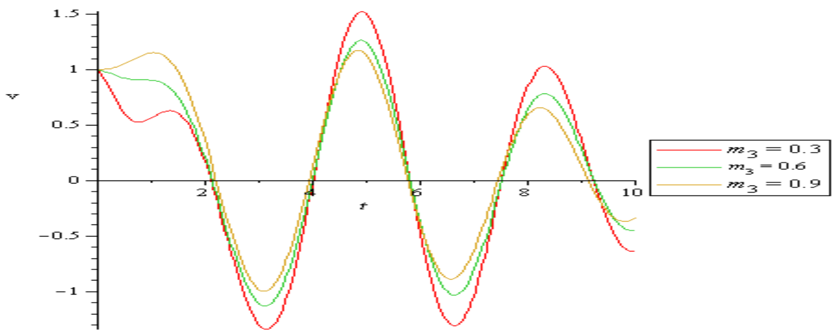

Fig 10 A graph of displacement *v against t*

Figs. 2,3 and 4 show that with increase in m_1, the amplitude of vibration in u reduces slightly while the amplitude of vibration of w and v increases slightly.

Figs. 5,6 and 7 show that with increase in m_2, the amplitude of vibration in w reduces slightly while the amplitude of vibration of u and v increases slightly.

Figs. 8,9 and 10 show that with increase in m_3, the amplitude of vibration in v and w reduces slightly while the amplitude of vibration of u increases slightly.

5.0 Conclusion

In conclusion, the V.I.M gives a good semi analytic approach of analyzing motions of masses while varying them against each other. It should be noted that the higher the mass at certain point, the lower the amplitude of vibration. It is observed that when a mass is of higher size than its other coupled masses, it reduces its amplitude of vibration, thereby increasing the amplitude of the other masses.

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