Journal of the Nigerian Association of Mathematical Physics
Volume **22 (November, 2012), pp 513 – 518**

On A Simple Procedure to Determine the Maximum Beam Deflection

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Abstract

A simplified procedure for determining the maximum beam deflection is studied. The problem was formulated using the differential equation of the deflection of a beam on elastic foundation subjected to any arbitrary transverse loads. The solution for a simply supported beam of variable moments of inertial subjected to end moments was obtained by matrix multiplication which was used to determine the segment where maximum deflection occur. The effects of foundation and flexural rigidity were also presented.

Keywords: simplified procedure, maximum, beam deflection, elastic foundation, matrix multiplication.

1.0 Introduction

The analysis of structures carrying loads, both static and dynamic is of considerable practical importance thereby provoking a lot of research activities. Bridges on which vehicles or trains travel and trolleys of overhead traveling cranes that moves on girders may be modeled as moving masses on simply supported beams. The middle of the last century witnessed substantial contributions of various scholars [1 - 6] to the development and improvement on the problem of deflection of bridges under traveling loads.

In 1959, Pei [7] carried out analysis of beams with variable moment of inertia using matrix multiplication approach. Contributions to the solution of this problem were made by Salvadori et al[8], Gilbert et al[9], and many others. Abubakar et al [10] investigated the reliability analysis of crane girders at ultimate limit state using matrix multiplication technique.

However, in this paper, we presented a simple procedure to determine the maximum deflection of beams. In addition to the works in [7] and [10], the effects of the elastic foundation and flexural rigidity on the response of simply supported beams were also presented in this analysis.

2.0 Formulation of the Problem

For a beam subjected to any arbitrary transverse loads, the differential equation of deflection in [10] is modified to include elastic foundation and expressed as:

$$\frac{d^2y}{dx^2} + ky = \frac{Mn}{EI_n} + \beta \qquad (2.1)$$

By dividing a beam into equal intervals of length h and designating equally spaced points, starting from left support such that expanding equation (2.1) by finite difference expression yields:

$$y_{n-1} + (kh^2 - 2)y_n + y_{n+1} = h^2\left(\frac{Mn}{EI_n} + \beta\right) \qquad (2.2)$$

For each pivotal point one equation can be written, so that a set of simultaneous linear equations equal to the number of unknown deflection points on the beam is obtained. The beam is divided into four equal segments with three equally speed pivotal points $\left(Y_{n-1,} Y_n \text{ and } Y_{n-1} \right)$ located between two and supports $\left(Y_{n-2,} Y_{n+2} \right)$.

Application of equation (2.2) in (2.1) result to simultaneous equations:

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$$Y_{n-2}+(kh^2-2)y_{n-1}+y_n=h^2\left(\frac{M_{n-1}}{EI_{n-1}}+\beta\right) \tag{2.3}$$

$$Y_{n-1}+(kh^2-2)y_n+y_{n+1}=h^2\left(\frac{M_n}{EI_n}+\beta\right) \tag{2.4}$$

$$Y_n+(kh^2-2)y_{n+1}+y_{n+2}=h^2\left(\frac{M_{n+1}}{EI_{n+1}}+\beta\right) \tag{2.5}$$

3.0 Method of Solutions

The governing equations for any beam of variable moment of inertia, subject to end moments and lateral loads which are applicable for various boundary conditions are solved using matrix multiplication to determine a segment where maximum deflection occurs. The simultaneous equations of equation (2.3), (2.4) and (2.5) can be written on compact matrix form as:

$$\{Y\}=[A]^{-1}\{C\} \tag{3.1}$$

Where (A) is a coefficient matrix and $(A)^{-1}$ is its inverse, (Y) is the deflection matrix, and (C) is the constant matrix. If there is no settlement for each end support, then

y_{n-2} and $y_{n+2}=0$

Thus for

$$[A]=\begin{bmatrix}(kh^2-2) & 1 & 0\\ 1 & (kh^2-2) & 1\\ 0 & 1 & (kh^2-2)\end{bmatrix},\quad \{Y\}=\begin{pmatrix}y_{n-1}\\ y_n\\ y_{n+1}\end{pmatrix},\quad \{C\}=h^2\begin{Bmatrix}\frac{M_{n-1}}{EI_{n-1}}+\beta\\ \frac{M_n}{EI_n}+\beta\\ \frac{M_{n+1}}{EI_{n+1}}+\beta\end{Bmatrix} \tag{3.2}$$

Solving for $[A]^{-1}=\dfrac{cof[A]^T}{|A|}$ (3.3)

Solving for $cof[A]$

$$A_{11}=\begin{vmatrix}(kh^2-2) & 1\\ 1 & (kh^2-2)\end{vmatrix}=(kh^2-2)^2-1,\quad A_{12}=\begin{vmatrix}1 & 1\\ 0 & (kh^2-2)\end{vmatrix}=-(kh^2-2)^2-1$$

$$A_{13}=\begin{vmatrix}1 & (kh^2-2)\\ 0 & 1\end{vmatrix}=1,\qquad A_{21}=\begin{vmatrix}1 & 0\\ 1 & (kh^2-2)\end{vmatrix}=(kh^2-2)^2$$

$$A_{23}=\begin{vmatrix}(kh^2-2) & 1\\ 0 & 1\end{vmatrix}=(kh^2-2),\quad A_{31}=\begin{vmatrix}1 & 0\\ (kh^2-2) & 1\end{vmatrix}=1$$

$$A_{31}=\begin{vmatrix}(kh^2-2) & 0\\ 1 & 1\end{vmatrix}=-(kh^2-2),\quad A_{33}=\begin{vmatrix}(kh^2-2) & 1\\ 1 & (kh^2-2)\end{vmatrix}=(kh^2-2)^2-1$$

$$cof(A)\begin{bmatrix}(Kh^2-2)^2-1 & -(Kh^2-2) & 1\\ -(Kh^2-2) & (Kh^2-2)^2 & -(Kh^2-2)\\ 1 & -(Kh^2-2) & (Kh^2-2)^2-1\end{bmatrix}$$

$$cof[A]^T=\begin{bmatrix}(kh^2-2)^2-1 & -(kh^2-2) & 1\\ -(kh^2-2) & (kh^2-2)^2 & -(kh^2-2)\\ 1 & -(kh^2-2) & (kh^2-2)^2-1\end{bmatrix} \tag{3.4}$$

From equation (3.2), we have:

$$\left| A \right| = (Kh^2-2)(Kh^2-2)^2-2 \tag{3.5}$$

By using (3.4) and (3.5) in (3.3), we obtained

$$[A]^{-1} = \frac{1}{\mu}\begin{bmatrix} (Kh^2-2)-\dfrac{1}{Kh^2-2)} & -1 & \dfrac{1}{kh^2-2} \\ -1 & (kh^2-2) & -1 \\ \dfrac{1}{\dfrac{1}{(kh^2-2)}} & -1 & (kh^2-2)-\dfrac{1}{kh^2-2} \end{bmatrix} \tag{3.6}$$

where $\mu = \left((kh^2-2)^2-2) \right)$

applying equations (3.2) and (3.6) in equation (3.1) yields

$$\begin{pmatrix} y_{n-1} \\ y_n \\ y_{n+1} \end{pmatrix} = \frac{h^2}{\mu}\begin{pmatrix} (kh^2-2)-\dfrac{1}{(kh^2-2)} & -1 & \dfrac{1}{(kh^2-2)} \\ -1 & (kh^2-2) & -1 \\ \dfrac{1}{(kh^2-2)} & -1 & (kh^2-2)-\dfrac{1}{(kh^2-2)} \end{pmatrix}\begin{pmatrix} \dfrac{M_{n-1}}{EI_{n-1}}+\beta \\ \dfrac{M_n}{EI_n}+\beta \\ \dfrac{M_{n+1}}{EI_{n+1}}+\beta \end{pmatrix}$$

where $\mu = \left((kh^2-2)^2-2) \right)$

such that

$$\begin{pmatrix} y_{n-1} \\ y_n \\ y_{n+1} \end{pmatrix} = \frac{h^2}{\mu}\begin{Bmatrix} H_1 \\ H_2 \\ H_3 \end{Bmatrix} \tag{3.7}$$

Where

$$H_1 = \frac{M_{n-1}(kh^2-2)}{EI_{n-1}} + \beta(kh^2-2) - \frac{-M_{n-1}}{EI_{n-1}} + -\frac{M_{n-1}}{EI_n} - \beta + \frac{M_{n+1}}{EI_{n+1}(kh^2-2)}$$

$$H_2 = \frac{-M_{n-1}}{EI_{n-1}} + \frac{M_{n-1}(kh^2-2)}{EI_n} + \beta(kh^2-2) - 2\beta$$

$$H_3 = \frac{M_{n-1}}{EI_{n-1}(kh^2-2)} - \frac{M_n}{EI_n} - \beta\frac{M_{n+1}(kh^2-2)}{EI_{n-1}} + \beta(kh^2-2) - \frac{M_{n+1}}{EI_{n+1}(kh^2-2)}$$

equation (3.7) has a merit of simplicity. After matrix multiplication, if it is found that the numerical value of y_{n-1} is the smallest among the three, then the maximum deflection occur in the interval $y_{n,}y_{n+1.}$ Then subdividing $y_{n,}$ y_{n+1} again into four equal subintervals of h^1 and designated pivotal points as $y^1{}_{n,}$ $y^1{}_n$ and $y_{n,}$ $y^1{}_{n+1}$. Using the same $[A]^{-1}$ but assigning different values into the elements of $[C]$, we get

$$\{C'\} = h^2 \left\{ \begin{matrix} \dfrac{M'_{n-1}}{EI_{n-1}} - Y_n + \beta \\ \dfrac{M'_n}{EI_n} + \beta \\ \dfrac{M'_{n+1}}{EI_{n+1}} - Y_{n+1} + \beta \end{matrix} \right\} \qquad (3.8)$$

Hence,

$$\begin{pmatrix} y_{n-1} \\ y_n \\ y_{n+1} \end{pmatrix} = \frac{h'^2}{\mu} \begin{pmatrix} (kh^2-2) - \dfrac{1}{(kh^2-2)} & -1 & \dfrac{1}{(kh^2-2)} \\ -1 & (kh^2-2) & -1 \\ \dfrac{1}{(kh^2-2)} & -1 & (kh^2-2) - \dfrac{1}{(kh^2-2)} \end{pmatrix} \begin{pmatrix} \dfrac{M'_{n-1}}{EI'_{n-1}} + y_n \\ \dfrac{M'_n}{EI'_n} + \beta \\ \dfrac{M'_{n+1}}{EI'_{n+1}} + y_{n+1} + \beta \end{pmatrix}$$

therefore,

$$\begin{pmatrix} y'_{n-1} \\ y'_n \\ y'_{n+1} \end{pmatrix} = \frac{h'^2}{\mu} \begin{pmatrix} X_1 \\ X_2 \\ X_{31} \end{pmatrix}$$

where

$$X_1 = \frac{M'_{n-1}(kh^2-2)}{EI'_{n-1}} + \beta(kh^2-2) - \frac{M'_{n-1}}{EI'_{n-1}(kh^2-2)} - \frac{M'_{n-1}}{EI'_{n-1}(kh^2-2)}$$

$$-\beta - Y_n(kh^2-2) + Y_n + \frac{M'_{n+1}}{EI'_{n-1}(kh^2-2)}$$

$$X_2 = \frac{M'_{n-1}(kh^2-2)}{EI'_n} + \beta(kh^2-2) - \frac{M'_{n+1}}{EI'_{n+1}} + y_n + y_{n+1} - \frac{M'_{n-1}}{EI'_{n-1}} - 2\beta$$

$$X_3 = \frac{M'_{n-1}}{EI'_{n-1}(kh^2-2)} - \frac{M'_{n-1}}{EI'^{1}_{n}} - + \beta(kh^2-2) - \frac{M'_{n+1}}{EI'_{n-1}(kh^2-2)} - \frac{Y_n}{(kh^2-2)}$$

$$-Y_{n+1}(kh^2-2) + \frac{Y_{n+1}}{(kh^2-2)}$$

4.0 Numerical Example

The maximum deflection can now be founded among y'_{n-1}, y'_n and y'_{n+1} values. The numerical calculations are carried out for a simply supported beam of variable moment of inertia, subject to end moments and lateral loads using the following values for computations. $M_1 = 32$, $M_2 = 80$, $M_3 = 68$, $h = 6ft$, $\beta = 0.5$, $K = 0.5$, $EI = 2770$, $I = 100$

The deflection points are $Y_1 = 1.0652$ $Y_2 = 0.9982$ $Y_3 = 1.0681$. The point $Y_3 = 1.0681$ is the maximum deflection.

However, the graphical representation of the maximum deflection of beams with different values of $k = 0.25, 0.5, 1.0, 1.5$ and 2.0 keeping other constant are shown in Fig 4.1

Table 4.1: Foundations (k) and maximum deflections (y)

Foundation (K)	Maximum Deflection (Y)
0.25	2.2979
0.5	2.2979
1.0	0.5172
1.5	0.3414
2.0	0.2548

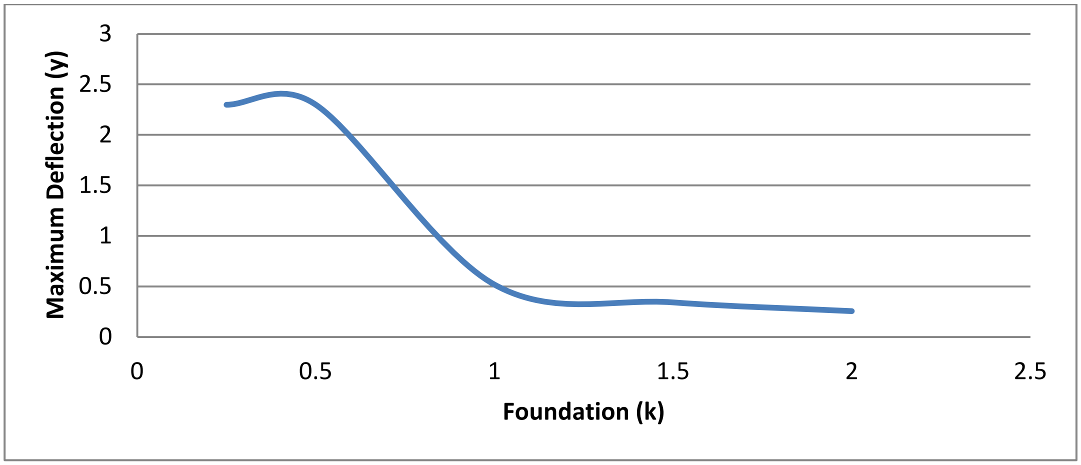

Fig 4.1 The Variation of the maximum deflections when foundation increases.

The graphical representation of the maximum deflection of a beams with different values of values of EI 240000000, 260000000, 270000000 and 280000000 consider other constant are shown in Fig 4.2

Table 4.2: The flexural rigidity and the maximum deflections

Flexural rigidity(EL)	Maximum Deflection (Y)
240000000	1.0636
260000000	1.0631
270000000	1.0629
280000000	1.0630

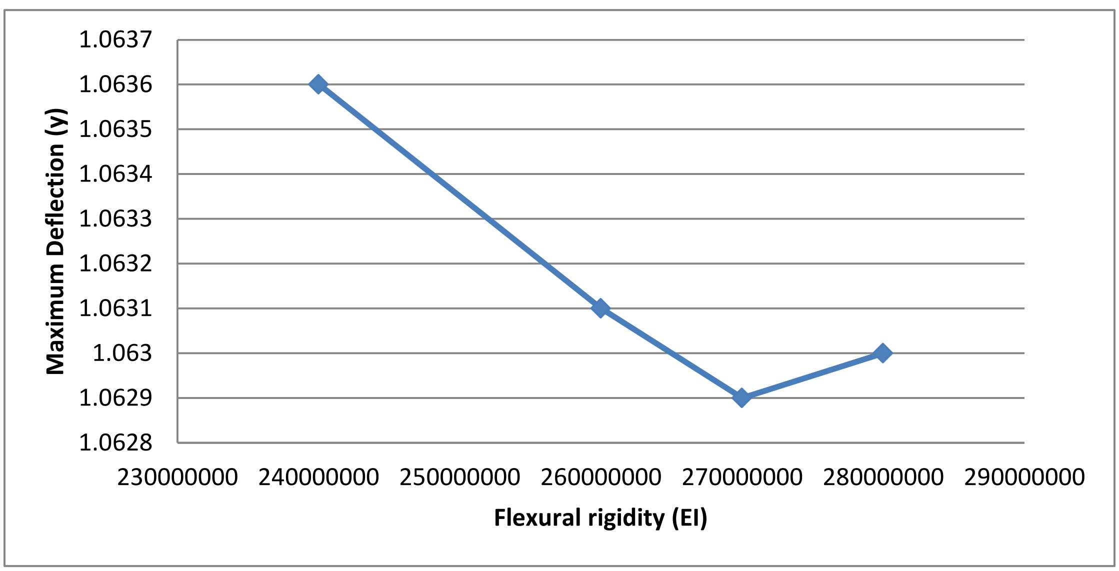

Fig 4.2 The Variation of the deflection of the beam by considering the fives values of EI

5.0 Conclusion

A simplified procedure for determining the maximum beam deflection is studied. The differential equations of deflection was expanded by finite difference and then transform into set of simultaneous linear equations equal to the number of unknown deflection point on the beam.

However, the solution of the differential equations of deflection is by solving the set of simultaneous linear equations which is equal to the number of unknown deflection point on the beam. The set of simultaneous linear equation is solved by transforming it into matrix form. In addition, there are no deflection at the end support, after matrix multiplication the highest value among the three unknown deflection point is the maximum deflection.

Finally, it was observed that the maximum beam deflection is decreasing as the foundation parameter K increases.

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