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On the Derivation of the Stability Function of a New Numerical Scheme for the Solution of Initial Value Problems in Ordinary Differential Equations.

[1]Ogunrinde R. B., and [2]Ibijola E.A.

Department of Mathematical Sciences,
University of Ado-Ekiti, Nigeria.

Abstract

This work presents the derivation and the analysis of the stability function of a new numerical scheme. It also compares the derived stability function with some existing ones. Stability is a very desirable property for any numerical integration algorithm, particularly if the initial value problem under consideration is to be stiff or stiff oscillatory.

Keywords: Stability Function, Ordinary differential equation, Numerical integrator.

1.0 Introduction

Any error introduced at any stage of computation can produce unstable numerical results if the problem under consideration is bad or the solution technique is bad. A numerical solution to a differential equation is unstable if as the procedure for its computation progresses, the numerical solutions of ordinary differential equations deviates significantly from the true solution. The peculiar nature of numerical solution of ordinary differential equation demands that we examine the stability properties of any newly derived schemes .We therefore consider in this paper the derivation of the stability polynomial of a new one - step scheme earlier derived by Ogunrinde [1] with a view to assessing its reliability. There are many excellent and exhaustive texts on this subject that may be consulted [2 – 7].

We shall investigate the stability properties using the well known A-stability model equation otherwise known as the test equations.

$$y^1 = l\, y,\ y(x_0) = y_0 \tag{1.0}$$

Suggested by [4] we shall also assume that $\mathrm{Re}(l\,) < 0$ and that the equation has a steady state solution.

$$y(x) = y(x_0)e^{l\,x} \tag{1.1}$$

Thus, a desired requirement of any integrator would be that the steady solution y(x) of the test equation agrees, as much as possible, with the solution of the associated difference equation which is the new scheme that approximates the differential equation.

Over the years, a large number of methods suitable for solving Ordinary Differential Equations (ODE) have been proposed. Generally the efficiency of any of the methods depends on the method's stability and certain accuracy properties. The accuracy properties of different methods are usually compared by considering the order of convergence as well as the truncation error coefficient of the various methods.

Ogunrinde [1] earlier proposed a numerical integration scheme or order six which is particularly well suited to solve initial value problem having oscillatory or exponential solutions. This method was based on the local representation of the theoretical solution y(x) to the initial value problem of the form

$$y' = f(x, y),\ y(a) = h \tag{1.3}$$

in the interval (x_t, x_{t+1}) by an polynomial interpolating function

$$F(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + b\,\mathrm{Re}(e^{rx+m}) \tag{1.4}$$

where a_0, a_1, a_2, a_3 *and* b are real undetermined coefficients, while r and m are complex parameters.

[1]Corresponding author: ***Ogunrinde R. B.,*** E-mail: -, Tel. +234 7031343029

2.0 The Basic Interpolant

Let us assume that the theoretical solution y(x) to the initial value problem (1.3) can be locally represented in the interval $(x_t, x_{t+1}), t \geq 0$ by the polynomial interpolating function (1.4). If we put,

$$r = r_1 + ir_2 \tag{1.5}$$

and $m = is$, $i^2 = -1$ in (1.4), we obtain the following interpolating function;

$$F(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + be^{r_1x}\cos(r_2x + s) \tag{1.6}$$

Let

$$R(x) = e^{r_1x} \tag{1.7}$$

and

$$q(x) = r_2x + s \tag{1.8}$$

Then, we obtain

$$F(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + bR(x)\cos q(x) \tag{1.9}$$

We shall assume that y_t is a numerical estimate to the theoretical solution $y(x_t)$ and that $f_t = f(x_t, y_t)$. We define mesh points as follows

$$x_t = a + th \tag{1.10}$$

t=0, 1, 2, 3…

With some imposed constraints, Ogunrinde [1] came out with a scheme of order six,

$$\begin{aligned}
y_{t+1} = y_t + \Bigg\{ & f_t - \left[f_t^{(1)} - \left(f_t^{(2)} - \frac{[(r_1^3 - 3r_1r_2^2)\cos q_t + (r_2^3 - 3r_1^2r_2)\sin q_t]f_t^{(3)}}{[(r_1^4 + r_2^4 - 6r_1^2r_2^2)\cos q_t + (4r_1r_2^3 - 4r_1^3r_2)\sin q_t]} \right) \right](a+th) \\
& - \frac{1}{2}\left[f_t^{(2)} - \frac{[(r_1^3 - 3r_1r_2^2)\cos q_t + (r_2^3 - 3r_1^2r_2\sin q_t)]f_t^{(3)}}{[(r_1^4 + r_2^4 - 6r_1^2r_2^2)\cos q_t + (4r_1r_2^3 - 4r_1^3r_2)\sin q_t]} \right](a+th) \\
& - \left[\frac{[r_1\cos q_t - r_2\sin q_t]f_t^{(3)}}{[(r_1^4 + r_2^4 - 6r_1^2r_2^2)\cos q_t + (4r_1r_2^3 - 4r_1^3r_2)\sin q_t]} \right] \Bigg\} h \\
+ \frac{1}{2}\Bigg\{ & f_t^{(1)} - \left[f_t^{(2)} - \frac{[(r_1^3 - 3r_1r_2^2)\cos q_t + (r_2^3 - 3r_1^2r_2)\sin q_t]f_t^{(3)}}{[(r_1^4 + r_2^4 - 6r_1^2r_2^2)\cos q_t + (4r_1r_2^3 - 4r_1^3r_2)\sin q_t]} \right](a+th) \\
& - \left[\frac{[(r_1^2 - r_2^2)\cos q_t - 2r_1r_2\sin q_t]f_t^{(3)}}{[(r_1^4 + r_2^4 - 6r_1^2r_2^2)\cos q_t + (4r_1r_2^3 - 4r_1^3r_2)\sin q_t]} \right] \Bigg\} (2ah + (1+2t)h^2) \\
+ \frac{1}{6}\Bigg\{ & \left[f_t^{(2)} - \frac{[(r_1^3 - 3r_1r_2^2)\cos q_t + (r_2^2 - 3r_1^2r_2)\sin q_t]f_t^{(3)}}{[(r_1^4 + r_2^4 - 6r_1^2r_2^2)\cos q_t + (4r_1r_2^3 - 4r_1^3r_2)\sin q_t]} \right] \\
& \left(3a^2h + ah^2(3+6t) + h^3(3t^2 + 3t + 1)\right) \\
& + \left[\frac{[e^{r_1h}(\cos q_t\cos r_2h - \sin q_t\sin r_2h) - \cos q_t]f_t^{(3)}}{[(r_1^4 + r_2^4 - 6r_1^2r_2^2)\cos q_t + (4r_1r_2^3 - 4r_1^3r_2)\sin q_t]} \right] \Bigg\}
\end{aligned} \tag{1.11}$$

3.0 Derivation of the Stability Function

To obtain the stability function for the new scheme (1.11), we proceed as follows;
Put

$$A_1 = \Bigg\{ f_t \left[f_t^{(1)} - \left(f_t^2 - \frac{[(\rho_1^3 - 3\rho_1\rho_2^2)\cos\theta_t + (\rho_2^3 - 3\rho_1^2\rho_2)\sin\theta_t] f_t^{(3)}}{[(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2)\cos\theta_t + (4\rho_1\rho_2^3 - 4\rho_1^3\rho_2)\sin\theta_t]} \right) \right] (\alpha + \tau h)$$
$$- \frac{1}{2}\left[f_t^{(2)} - \frac{[(\rho_1^3 - 3\rho_1\rho_2^2)\cos\theta_t + (\rho_2^3 - 3\rho_1^2\rho_2)\sin\theta_t] f_t^{(3)} (\alpha + \tau h)^2}{[(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2)\cos\theta_t + (4\rho_1\rho_2^3 - 4\rho_1^3\rho_2)\sin\theta_t]} \right.$$
$$\left. + \frac{(\rho_1\cos\theta_t - \rho_2\sin\theta_t) f_t^{(3)}}{[(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2)\cos\theta_t + (4\rho_1\rho_2^3 - 4\rho_1^3\rho_2)\sin\theta_t]} \right] h \Bigg\} \tag{1.12}$$

$$A_2 = \frac{1}{2}\Bigg\{ f^{(1)}_t - \left[f_t^{(2)} - \frac{[(\rho_1^3 - 3\rho_1\rho_2^2)\cos\theta_t + (\rho_2^3 - 3\rho_1^2\rho_2)\sin\theta_t] f_t^{(3)} (\alpha + \tau h)}{[(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2)\cos\theta_t + (4\rho_1\rho_2^3 - 4\rho_1^3\rho_2)\sin\theta_t]} \right]$$
$$- \left[\frac{[(\rho_1^2 - \rho_2^2)\cos\theta_t - 2\rho_1\rho_2\sin\theta_t] f_t^{(3)}}{[(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2)\cos\theta_t + (4\rho_1\rho_2^3 - 4\rho_1^3\rho_2)\sin\theta_t]} \right] \Bigg\} (2\alpha h + (1 + 2\tau)h^2$$

$$A_3 = \frac{1}{6}\Bigg\{ \left[f_t^{(2)} - \frac{[(\rho_1^3 - 3\rho_1\rho_2^2)\cos\theta_t + (\rho_2^3 - 3\rho_1^2\rho_2)\sin\theta_t] f_t^{(3)}}{[(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2)\cos\theta_t + (4\rho_1\rho_2^3 - 4\rho_1^3\rho_2)\sin\theta_t]} \right]$$
$$(3\alpha^2 h + \alpha h^2(3 + 6\tau) + h^3(3\tau^2 + 3\tau + 1)\Bigg\}$$

$$A_4 = \left[\frac{(e^{\rho_1 h}(\cos\theta_t \cos\rho_2 h - \sin\theta_t \sin\rho_2 h) - \cos\theta_t) f_t^{(3)}}{(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2)\cos\theta_t + (4\rho_1\rho_2^3 - 4\rho_1^3\rho_2)\sin\theta_t} \right]$$

We proceed to expand the above terms with the fact that

Using the Maclaurin series of $e^{\rho_1 h}, \cos\rho_2 h$ and $\sin\rho_2 h$ respectively, then
On simplification, equation (1.11) yields

$$y_{t+1} = y_t + \lambda h y_t + \frac{1}{2}h^2\lambda^2 y_t + \frac{1}{6}h^3\lambda^3 y_t$$
$$\Rightarrow \; y_{t+1} = y_t(1 + \lambda h + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3) \tag{1.13}$$
$$\frac{y_{t+1}}{y_t} = (1 + \lambda h + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3)$$

We define $z = \lambda h$
We have

$$\mu(z) = \frac{y_{t+1}}{y_t} = (1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3)$$

Therefore

$$\mu(z) = \left[1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 \right] \tag{1.14}$$

Equation (1.14) is the derived stability function for the scheme derived in [1].

Fatunla [8] had earlier proposed a numerical integration scheme. His method was based on the local representation of the theoretical solution y(x) to the initial value problem of the form $y' = f(x, y),\; y(a) = \eta$ in the interval (x_n, x_{n+1}) by polynomial interpolating function $F(x) = a_0 + a_1 x + b\,\mathrm{Re}(e^{(rx+\mu)})$, where a_0, a_1 *and* b are real undetermined coefficients, while r and μ are complex parameters. He was able to derive the stability function as

$$m(z) = [1 + z] \tag{1.15}$$

Ibijola [7] Improved on [2] by proposing a new numerical integration scheme also suited for initial value problems of the form $y' = f(x, y),\ y(a) = h$

In the interval (x_n, x_{n+1}) by a polynomial interpolating function $F(x) = a_0 + a_1 x + a_2 x^2 + b\,\mathrm{Re}(e^{rx+m})$ where a_0, a_1, a_2 *and* b are real undetermined coefficients and r and m are complex parameters.

The work of Ibijola [7] is an improvement over the work in [8] because, Ibijola's work is of order five while Fatunla's work [8] is of order four. Ibijola [7] also derived the stability function as

$$m(z) = \left[1 + z + \frac{1}{2}z^2\right] \tag{1.16}$$

Definition 1.1

The One step scheme (1.11) is said to be absolutely stable at a point Z in the complex plane provided the stability function or polynomial function $m(z)$ fulfils the following conditions

$$|m(z)| < 1$$

And the region of absolute stability is defined as

$$RAS = \{z : |m(z)| < |\} \tag{1.17}$$

Definition 1.2

The numerical integration scheme (1.11) is said to be A-Stable provided the region of absolute stability specified in (1.15) includes the entire left half of the complex Z-plane.

A-stability is a very desirable property for any numerical integration algorithm, particularly if the IVP (1.13) was to be stiff or stiff oscillatory (i.e. and inherently stable differential system (1.13) in which the interval of integration is large).

4.0 Conclusion

Comparing the Stability functions in the works of [7, 8] and the newly derived one, it is obvious that the new stability polynomial is an improvement on the previous works [7, 8]. With stability polynomial of higher degree in the newly derived stability function we have an assurance that the scheme developed will be more stable and reliable.

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