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Formulation of a Symmetric Implicit Super Runge-Kutta Nyström Method for the Integration of Third Order ODEs

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Abstract

In this paper, we extend of the Symmetric Implicit Runge-Kutta Method for the integration of first order ODEs to a Symmetric Implicit Super Runge-Kutta Nyström Method for Direct Integration of General Third Order initial value problems (IVPs).The theory of Nyström method was adopted in the derivation of the method. The method has an implicit structure for efficient implementation and produces simultaneously approximation of the solution of general Third Order IVPs. The proposed method was tested with Numerical experiment to illustrate its efficiency and the method can be extended to solve higher order differential equations.

Keywords: Symmetric Method, Implicit Runge-Kutta Nyström Method, The theory of Nyström Method, General third order IVPs.

1.0 Introduction

There is a vast body of literature addressing the numerical solution of the initial value problems (IVPs) of the form

$$y'' = f(x,y) \qquad y(x_o) = y \qquad y'(x_o) = \beta \tag{1.1}$$

$$y'' = f(x,y,y') \qquad y(x_o) = y \qquad y'(x_o) = \beta \tag{1.2}$$

$$y''' = f(x,y) \qquad y(x_o) = y \qquad y'(x_o) = \beta \qquad y''(x_o) = \alpha \tag{1.3}$$

(see for example[1],[2],[3] and[4]) but not so much for the special third order IVP of the form

$$y''' = f(x,y,y',y'') \qquad y(x_o) = y \qquad y'(x_o) = \beta \qquad y''(x_o) = \alpha \tag{1.4}$$

(Different approaches appear in [5] and [6]).

Although it is possible to integrate a third order IVP by reducing it to first order system and apply one of the method available for such system it seem more natural to provide commercial method in order to integrate the problem directly. The advantage of these approaches lies in the fact that they are able to exploit special information about ODEs and this result in an increase in efficiency (that is, high accuracy at low cost). For instance, it is well known that Runge-Kutta Nyström method for (1.2) involve a real improvement as compared to standard Runge-Kutta method for a given number of stages [7, p.285].

In this paper,we present a five stage Symmetric Implicit Super Runge-Kutta Nyström Method for Direct Integration of General Third Order IVPs with the following advantage such as high order and stage order, low error constant and low implementation cost.

For the first order differential equations

$$y' = f(x,y) \qquad y(x_o) = y \tag{1.5}$$

[8,9] defined an s-stage implicit Runge-Kutta method in the form

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$$y_{n+1} = y_n + h\sum_{i=1}^{s} w_i k_i \qquad (1.6)$$

Where for i = 1, 2 - - - - s.

$$K_i = f(x_i + \alpha_j h, y_n + h\sum_{j=1}^{s} a_{ij} k_j) \qquad (1.7)$$

The real parameters α_j, k_i, a_{ij} define the method.

An s-stage implicit Runge-kutta Nystrom for direct integration of general second order IVP (1.1 and 1.2) is defined in the form

$$y_{n+1} = y_n + \alpha_i h y'_n + h^2\sum_{j=1}^{i-1} a_{ij} k_j \qquad (1.8)$$

$$y'_{n+1} = y'_n + h\sum_{j=1}^{i-1} \bar{a}_{ij} k_j \qquad (1.9)$$

Where for i = 1, 2 - - - - s.

$$K_i = f(x_i + \alpha_j h, y_n + \alpha_i y'_n + h^2\sum_{j=1}^{i-1} a_{ij} k_j, y'_n + h\sum_{j=1}^{i-1} \bar{a}_{ij} k_j) \qquad (2.0)$$

The real parameters $\alpha_j, k_i, a_{ij}, \bar{a}_{ij}$ define the method [10,11]

Based on (1.6-2.0) and Taylor series expansion

$$y(x+h) = y(x) + \frac{h}{!}y'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \ldots + \frac{h^k}{k!}y^k + 0(h)^{k+1}$$

$0(h)^{k+1}$ is the local errors. As $k \to \infty$, $0(h)^{k+1} \to 0$, since $0 < h < 1$.

We proposed an s-stage implicit Runge-Kutta for direct integration of third order IVP (1.3 and 1.4) as defined in the form

$$y_{n+1} = y_n + \alpha_i h y'_n + \frac{(\alpha_i h)^2}{2} y''_n + h^3\sum_{j=1}^{i-1} a_{ij} k_j \qquad (2.1)$$

$$y'_{n+1} = y'_n + \alpha_i h y''_n + h^2\sum_{j=1}^{i-1} \bar{a}_{ij} k_j \qquad (2.2)$$

$$y''_{n+1} = y''_n + h\sum_{j=1}^{i-1} \bar{\bar{a}}_{ij} k_j \qquad (2.3)$$

Where for i = 1, 2 - - - - s.

$$K_i = f(x_i + \alpha_j h, y_n + \alpha_i y'_n + \frac{(\alpha_i h)^2}{2} y'' + h^3\sum_{j=1}^{i-1} a_{ij} k_j y'_n + \alpha_i{}^2 h y''_n + + h^2\sum_{j=1}^{i-1} \bar{a}_{ij} k_j, y''_n + h\sum_{j=1}^{i-1} \bar{\bar{a}}_{ij} k_j) \qquad (2.4)$$

The real parameters $\alpha_j, k_i, a_{ij}, \bar{a}_{ij}, \bar{\bar{a}}_{ij}$ define the method

The paper is organized as follows: In section 2 we will show how the butcher's implicit Runge-Kutta methods for the first order differential equations tableau are modified to include second (that is implicit Runge-Kutta Nystrom method) and which we extended to third derivatives. In section 3 we offer the main derivation of the Symmetric Implicit Super Runge-Kutta Nyström Method for Direct Integration of General Third Order IVPs.Finally,some numerical experiments are presented in section 4.

2.0 Butcher's implicit Runge-kutta methods for the first order differential equations

The method (1.6) in Butcher-array form can be written as

$$\begin{array}{c|c} \alpha & \beta \\ \hline & W^T \end{array} \qquad (2.5)$$

Where matrix α and $\beta = a_{ij}$ are obtained in (1.7)

While for the implicit Runge–Kutta method for the numerical integration of the second order initial value problem (1.1) and (1.2) the method (1.8) in Butcher – array, form is.

$$\begin{array}{c|c|c} \alpha & \bar{A} & A \\ \hline & \bar{b}^T & b \end{array}$$

A = a_{ij} = β^2 , Ā = $\bar{a}_{ij}$ = β

β = βe , $\bar{b}$ = W , b = $W^T\beta$ ………(2.6)

(see [11])

In Butcher – array form our proposed method for (1.3) and 1.4) is

$$\begin{array}{c|c|c|c} \alpha & \bar{\bar{A}} & \bar{A} & A \\ \hline & \bar{\bar{b}}^T & \bar{b}^T & b \end{array}$$

A= a_{ij} = β^3 Ā = $\bar{a}_{ij}$ = β^2 $\bar{\bar{A}}$ = $\bar{\bar{a}}_{ij}$ = β

β = βe , $\bar{b}$ = W $\bar{b}^T$ = $W^T\beta$ $\bar{\bar{b}}^T$ = $W^T\beta^2$ (2,7)

3.0 Derivation of the Method

We particularly wish to emphasize the combination of a multi-step structure with the use of off-step points, we seek a method that are multistage and multi-value because it will be convenient to extend the general linear method formulation to the high order Runge – Kutta case [10] by Considering the Symmetric Implicit Runge-Kutta Nyström Method for the Integration of General Second Order ODEs [1] given by

$$k_1 = f(x_n, y_n, y'_n)$$

$$k_2 = f(x_n + \frac{1}{4}h, y_n + \frac{1}{4}hy'_n + h^2(\frac{17}{1152}k_1 + \frac{9}{320}k_2 - \frac{37}{1920}k_3 + \frac{7}{720}k_4 - \frac{1}{480}k_5),$$
$$y'_n + h(\frac{251}{2880}k_1 + \frac{323}{1440}k_2 - \frac{11}{120}k_3 + \frac{53}{1440}k_4 - \frac{19}{2880}k_5))$$

$$k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hy'_n + h^2(\frac{13}{360}k_1 + \frac{37}{360}k_2 - \frac{1}{40}k_3 + \frac{1}{72}k_4 - \frac{1}{360}k_5),$$
$$y'_n + h(\frac{29}{360}k_1 + \frac{31}{90}k_2 + \frac{1}{15}k_3 + \frac{1}{90}k_4 - \frac{1}{360}k_5))$$

$$k_4 = f(x_n + \frac{3}{4}h, y_n + \frac{3}{4}hy'_n + h^2(\frac{9}{160}k_1 + \frac{3}{16}k_2 + \frac{9}{640}k_3 + \frac{9}{320}k_4 - \frac{3}{640}k_5),$$
$$y'_n + h(\frac{27}{320}k_1 + \frac{51}{160}k_2 + \frac{9}{40}k_3 + \frac{21}{160}k_4 - \frac{3}{320}k_5))$$

$$k_5 = f(x_n + h, y_n + hy'_n + h^2(\frac{7}{90}k_1 + \frac{4}{15}k_2 + \frac{1}{15}k_3 + \frac{4}{45}k_4 + 0k_5),$$
$$y'_n + h(\frac{7}{90}k_1 + \frac{16}{45}k_2 + \frac{2}{15}k_3 + \frac{16}{45}k_4 + \frac{7}{90}k_5))$$

$$y_{n+1} = y_n + hy'_n + h^2(\frac{7}{90}k_1 + \frac{4}{15}k_2 + \frac{1}{15}k_3 + \frac{4}{45}k_4 + 0k_5)$$
$$\mathrm{y}'_{\mathrm{n+1}} = y'_n + h(\frac{7}{90}k_1 + \frac{16}{45}k_2 + \frac{2}{15}k_3 + \frac{16}{45}k_4 + \frac{7}{90}k_5) \quad \ldots(2.8)$$

Expressing (2.8) in the form (2.6) we have the matrices $\alpha_j, A = a_{ij}, \bar{A} = \bar{a}_{ij}, b^T, \bar{b}^T$ as

$$
\begin{array}{c|ccccc|ccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{251}{2880} & \frac{323}{1440} & \frac{-11}{120} & \frac{53}{1440} & \frac{-19}{2880} & \frac{17}{1152} & \frac{9}{320} & \frac{-37}{1920} & \frac{7}{720} & \frac{-1}{480} \\
\frac{1}{2} & \frac{29}{360} & \frac{31}{90} & \frac{1}{15} & \frac{1}{90} & \frac{-1}{360} & \frac{13}{360} & \frac{37}{360} & -\frac{1}{40} & \frac{1}{72} & \frac{-1}{360} \\
\frac{3}{4} & \frac{27}{320} & \frac{51}{160} & \frac{9}{40} & \frac{21}{160} & \frac{-3}{320} & \frac{9}{160} & \frac{3}{16} & \frac{9}{640} & \frac{9}{320} & \frac{-3}{640} \\
1 & \frac{7}{90} & \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} & \frac{7}{90} & \frac{4}{15} & \frac{1}{15} & \frac{4}{45} & 0 \\
\hline
 & \frac{7}{90} & \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} & \frac{7}{90} & \frac{4}{15} & \frac{1}{15} & \frac{4}{45} & 0
\end{array}
\qquad \ldots(2.9)
$$

As characterized by the theory of Nystôm method (see[10] and [11]).

Using equation (2.7) we obtained $\alpha_j, A = a_{ij}, \overline{A} = \overline{a}_{ij}, \overline{\overline{A}} = \overline{\overline{a}}_{ij}, b^T, \overline{b}^T, \overline{\overline{b}}^T$ as

$$
\begin{array}{c|ccccc|ccccc|ccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{251}{2880} & \frac{323}{1440} & \frac{-11}{120} & \frac{53}{1440} & \frac{-19}{2880} & \frac{17}{1152} & \frac{9}{320} & \frac{-37}{1920} & \frac{7}{720} & \frac{-1}{480} & \frac{287}{184320} & \frac{187}{92160} & \frac{-1}{512} & \frac{25}{18432} & \frac{-71}{184320} \\
\frac{1}{2} & \frac{29}{360} & \frac{31}{90} & \frac{1}{15} & \frac{1}{90} & \frac{-1}{360} & \frac{13}{360} & \frac{37}{360} & -\frac{1}{40} & \frac{1}{72} & \frac{-1}{360} & \frac{91}{11520} & \frac{103}{5760} & \frac{-1}{120} & \frac{5}{1152} & \frac{-11}{11520} \\
\frac{3}{4} & \frac{27}{320} & \frac{51}{160} & \frac{9}{40} & \frac{21}{160} & \frac{-3}{320} & \frac{9}{160} & \frac{3}{16} & \frac{9}{640} & \frac{9}{320} & \frac{-3}{640} & \frac{399}{20480} & \frac{111}{2048} & \frac{-27}{2560} & \frac{93}{10240} & \frac{-39}{20480} \\
1 & \frac{7}{90} & \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} & \frac{7}{90} & \frac{4}{15} & \frac{1}{15} & \frac{4}{45} & 0 & \frac{13}{360} & \frac{1}{9} & 0 & \frac{1}{45} & \frac{-1}{360} \\
\hline
 & \frac{7}{90} & \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} & \frac{7}{90} & \frac{4}{15} & \frac{1}{15} & \frac{4}{45} & 0 & \frac{13}{360} & \frac{1}{9} & 0 & \frac{1}{45} & \frac{-1}{360}
\end{array}
$$

…(3.0)

Putting (3.0) in (2.4) give the Symmetric Implicit Super Runge-Kutta Nyström Method(SISRKN) for Direct Integration of General Third Order initial value problems(IVPs):

$$k_1 = f(x_n, y_n, y'_n, y''_n)$$

$$k_2 = f(x_n + \frac{1}{4}h, y_n + \frac{1}{4}hy'_n + \frac{(\frac{1}{4}h)^2}{2}y''_n + h^3(\frac{287}{184320}k_1 + \frac{187}{92160}k_2 - \frac{1}{512}k_3 + \frac{25}{18432}k_4 - \frac{71}{184320}k_5),$$
$$y'_n + \frac{1}{4}hy''_n + h^2(\frac{17}{1152}k_1 + \frac{9}{320}k_2 - \frac{37}{1920}k_3 + \frac{7}{720}k_4 - \frac{1}{480}k_5),$$
$$y''_n + h(\frac{251}{2880}k_1 + \frac{323}{1440}k_2 - \frac{11}{120}k_3 + \frac{53}{1440}k_4 - \frac{19}{2880}k_5))$$
$$k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hy'_n + \frac{(\frac{1}{2}h)^2}{2}y''_n + h^3(\frac{91}{11520}k_1 + \frac{103}{5760}k_2 - \frac{1}{120}k_3 + \frac{5}{1152}k_4 - \frac{11}{11520}k_5),$$
$$y'_n + \frac{1}{2}hy''_n + h^2(\frac{13}{360}k_1 + \frac{37}{360}k_2 - \frac{1}{40}k_3 + \frac{1}{72}k_4 - \frac{1}{360}k_5),$$
$$y''_n + h(\frac{29}{360}k_1 + \frac{31}{90}k_2 + \frac{1}{15}k_3 + \frac{1}{90}k_4 - \frac{1}{360}k_5))$$
$$k_4 = f(x_n + \frac{3}{4}h, y_n + \frac{3}{4}hy'_n + \frac{(\frac{3}{4}h)^2}{2}y''_n + h^3(\frac{399}{20480}k_1 + \frac{111}{2048}k_2 - \frac{27}{2560}k_3 + \frac{93}{10240}k_4 - \frac{39}{20480}k_5),$$
$$y'_n + \frac{3}{4}hy''_n + h^2(\frac{9}{160}k_1 + \frac{3}{16}k_2 + \frac{9}{640}k_3 + \frac{9}{320}k_4 - \frac{3}{640}k_5),$$
$$y''_n + h(\frac{27}{320}k_1 + \frac{51}{160}k_2 + \frac{9}{40}k_3 + \frac{21}{160}k_4 - \frac{3}{320}k_5))$$
$$k_5 = f(x_n + h, y_n + hy'_n + \frac{(h)^2}{2}y''_n + h^3(\frac{13}{360}k_1 + \frac{1}{9}k_2 + 0k_3 + \frac{1}{45}k_4 - \frac{1}{320}k_5),$$
$$y'_n + hy''_n + h^2(\frac{7}{90}k_1 + \frac{4}{15}k_2 + \frac{1}{15}k_3 + \frac{4}{45}k_4 + 0k_5),$$
$$y''_n + h(\frac{7}{90}k_1 + \frac{16}{45}k_2 + \frac{2}{15}k_3 + \frac{16}{45}k_4 + \frac{7}{90}k_5))$$
$$y_{n+1} = y_n + hy'_n + \frac{(h)^2}{2}y''_n + h^3(\frac{13}{360}k_1 + \frac{1}{9}k_2 + 0k_3 + \frac{1}{45}k_4 - \frac{1}{320}k_5)$$
$$y'_{n+1} = y'_n + hy''_n + h^2(\frac{7}{90}k_1 + \frac{4}{15}k_2 + \frac{1}{15}k_3 + \frac{4}{45}k_4 + 0k_5) \qquad \ldots(3.1)$$
$$y''_{n+1} = y''_n + h(\frac{7}{90}k_1 + \frac{16}{45}k_2 + \frac{2}{15}k_3 + \frac{16}{45}k_4 + \frac{7}{90}k_5)$$

4. 0 Numerical Experiment

To study the efficiency of method we present some numerical examples widely used by several authors such as [3] and [6].The linear Multistep Method(LMM) was used by [3] while Modified fourth-order Runge-Kutta Method(RK) was used by[6]. Using (3.1) the SISRKN method for direct integration of third order ODEs to solve these problems, the absolute errors (i.e. absolute values of the theoretical solution minus approximate solutions) were compared in Table1 and Table 2.

Problem 1 [3]

$y''' = 3\cos x$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 2$, h=0.1, $0 \le x \le 0.5$

Theoretical Solution: $y(x) = x^2 + 3x + 1 - 3Sinx$

Problem 2 [6]

$y''' + 3y'' + 3y' + y = e^{-x}\sin x$ $y(0) = 2$, $y'(0) = 0$, $y''(0) = -1$ h=0.1, $0 \le x \le 0.5$

Theoretical Solution: $y(x) = e^{-x}(1+x)^2 + \cos x$

Table 1: Absolute errors of Problem 1

x	LMM[3]	SISRKN
0.1	1.04E-06	5.95.E+11
0.2	5.06E-06	3.85.E-10
0.3	1.21E-05	1.60.E-11
0.4	3.53E-05	7.40.E-11
0.5	5.15E-05	1.87E-10

Table 2: Absolute errors of Problem 2

x	RK[6]	SISRKN
0.1	3.53.E-07	3.31E-10
0.2	3.17E-07	7.75E-10
0.3	5.47E-07	9.78E-10
0.4	4.86E-07	1.31E-10

Problem 3

The ordinary differential equation of the form $y''' = y^{-2}$ has been derived by Tanner [12] to investigate the motion of the contact line for a thin oil drop spreading on a horizontal surface.

Momoniat and Mahomed[5], reduced the third-order ODE above by successive reduction of order to a second-order ODE and then to a first-order ODE.A fourth-order Runge-Kutta method to solve the first-order ODE $y' = f(x; y)$. An initial value $y(0)$ is chosen with successive values determined by Symmetry Reduction and Numerical Solution of a Third-Order ODE. Using the SISRKN for direct integration of third order ODEs to solve this problem taking $h = 0.1$ gives;

Table 3 Comparing numerical values of the numerical solution obtained using a fourth-order Runge-Kutta method, Symmetry Reduction of a Third-OrderODEand the Super Runge-Kutta Nyström method for direct integration of third order ODEs at $x \in [0; 0.2; 0.4; 0.6; 0.8; 1.0]$ taking $h = 0.01$ and $k = 2$ for the initial conditions $y(0) = y'(0) = y''(0) = 1$.

X	Exact Solution	RK4 Method	Symmetry Reduction[5]	SISRKNMethod
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.2	1.221211030	1.221210005	1.221210004	1.221210005
0.4	1.488834893	1.488834780	1.488834779	1.4888834780
0.6	1.807361404	1.807361398	1.807361397	1.807361398
0.8	2.179819234	2.179819234	2.179819233	2.179819234
1.0	2.608275822	2.608274868	2.608274867	2.608274868

5.0 Conclusion

Through the approach presented in this paper, the RK method can be extended to solve higher order differential equations. The method requires less work with very little cost (when compared with classical and improved RK) and possesses a gain in efficiency (when compared with LMM). The method is self starting with no overlapping of solution models.

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