Journal of the Nigerian Association of Mathematical Physics
***Volume* 22 (November, 2012), pp 419 – 424**
© *J. of NAMP*

High Order Quasi - Runge - Kutta Methods By Refinement Process For The Solution Of Initial Value Problems.

Salisu Abdullahi And Adeboye K.R

Departmentof Mathematics and Computer Science
Niger State Polytechnic Zungeru,
Niger State, Nigeria

Abstract

In this paper, we derive some Quasi-RungeKuttaMethods, through a refinement process, which have better approximation and less computational steps. The new schemes are consistent, zero-stable and convergent. Also, provided is an example of initial value problem solved with the new schemes and the results help to establish their degree of accuracy and efficiency.

1.0 Introduction

The use of simple operations to find approximate solutions to complex problems constitutes the main goal of numerical analysis. One of the major tasks of numerical analysis is that of solving differential equations. Solutions to differential equations are obtained by using analytical or numerical methods. Those solutions are often useful as they provide excellent insight into the behavior of some systems. However, analytical solutions can be obtained for only a limited class of problems. These include those that can be approximated with linear models and those that have simple geometry and low dimensionality. Consequently, analytical solutions are of limited practical value because most real-life problems are non-linear and involve complex shapes and processes.

In such cases, where differential equations defy solutions analytically, approximate solutions are often obtainable by the application of numerical methods. It is well known that initial valued problems of ordinary differential equations often arise in many practical applications, such as chemical reactor, theory of fluid mechanics, automatic control and combustion e.t.c [1-2]. The traditional methods for solving ODEs generally fall into two main classes: linear multistep and Runge-Kutta methods [3-4]. Various reasons determine the choice of one method over another, two obvious criteria being speed and accuracy. However, the advent of fast and efficient digital computers has increased the role of numerical methods in solving scientific, engineering as well as social problems.[5]

2.0 The Refinement Process

We consider the mid point finite difference method:

$$\therefore y_{n+2} = y_n + 2hf_{n+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We expand y_{n+2} and f_{n+1} to find the Error term:

$$\therefore y_n + 2hy_n^1 + \frac{4h^2}{2}y_n^{11} + \frac{8h^3}{6}y_n^{111} + ----y_n - 2h\left[f_n + hf_n^1 + \frac{h^2}{2}f_n^{11}\right]$$

$$=\rangle y_n + 2hy_n^1 + 2h^2y_n^{11} + \frac{4}{3}h^3y_n^{111} --- y_n - 2hf_n - 2h^2f_n^1 - h^3f_n^{11}$$

$$=\left[\frac{4h^3}{3} - h^3\right]f_n^{11} =\rangle \frac{h^3}{3}f_n^{11}$$

*Corresponding author: ***Salisu Abdullahi,*** E-mail: saddullah2k3@yahoo.com, Tel. +234 8035925783

$$\therefore The\, Error\, term\, is\ \frac{h^3}{3} f_n^{11}$$

$$\frac{h^3}{12} f_n^{11} = \frac{h^3}{3}\left[\frac{f_{n+1/2}^1 - f_n^1}{h/2}\right] = \frac{2h^2}{3}\left[f_{n+1/2}^1 + f_n^1\right]$$

$$\frac{4h}{3}\left[f_{n+1} + f_{n+1/2}\right] - \frac{2h}{3}\left[f_{n+1} - f_n\right]$$

$$= \frac{h}{3}\left[2 f_{n+1} + 4 f_{n+1/2} + 2 f_n\right] \text{.................................}(2)$$

We combine equations (2) and (1) above to obtain

$$\therefore y_{n+2} = y_n + 2hf_{n+1} + \frac{h}{3}\left[2 f_{n+1} - 4 f_{n+1/2} + 2 f_n\right]$$

$$\therefore y_{n+2} = y_n + \frac{h}{3}\left[8 f_{n+1} - 4 f_{n+1/2} + 2 f_n\right] \text{........................}(\textbf{Scheme1})$$

Again,we consider

$$y_{n+2} = y_n + 2hf_{n+1} + \frac{h^3}{3} f_n^{11}$$

$$\frac{h^3}{3} f_n^{11} = \frac{h^3}{3}\left[\frac{f_{n+1} - 2 f_n + f_{n-1}}{h^2}\right]$$

$$= \frac{h}{3}\left[f_{n+1} - 2 f_n + f_{n-1}\right] \text{................................}(3)$$

Combining equations (3) and (1) above, we have :

$$y_{n+2} = y_n + 2hf_{n+1} + \frac{h}{3}\left[f_{n+1} - 2 f_n + f_{n-1}\right]$$

$$\Rightarrow y_{n+2} = y_n + \frac{h}{3}\left[7 f_{n+1} - 2 f_n + f_{n-1}\right] \text{..........................}(\textbf{Scheme2})$$

We consider the scheme (2 step Adam-Bashforth method):

$$y_{n+2} = y_{n+1} + \frac{h}{2}\left[3 f_{n+1} - f_n\right] \text{...................................}(4)$$

The Error term is

$$y_{n+2} - y_{n+1} - \frac{h}{2}\left[3 f_{n+1} - f_n\right] \text{...................................}(5)$$

We expand y_{n+2}, y_{n+1} and f_{n+1} in Taylor's series

i.e

$$y_n + 2hy_n^1 + 2h^2 y_n^{11} + \frac{4}{3}h^3 y_n^{111} - y_n - hy_n^1 - \frac{h^2}{2} y_n^{11} - \frac{h^3}{6} y_n^{111} - \frac{3h}{2}\left[f_n + hf_n^1 + \frac{h}{2} f_n^{11}\right] + \frac{h}{2} f_n$$

$$\Rightarrow y_n + 2hy_n^1 + 2h^2 y_n^{11} + \frac{4}{3}h^3 y_n^{111} - y_n - hy_n^1 - \frac{h^2}{2} y_n^{11} - \frac{h^3}{6} y_n^{111} - \frac{3h}{2} f_n - \frac{3h^2}{2} f_n^1 - \frac{3h^3}{4} f_n^{11} + \frac{h}{2} f_n$$

$$\therefore \left[\frac{4h^3}{3} - \frac{h^3}{6}\right] y_n^{111} - \frac{3h^3}{4} f_n^{11} = \frac{5h^3}{12} f_n^{11}$$

The Error term is $\dfrac{5h^3}{12} f_n^{11}$

$$\frac{5h^3}{12} f_n^{11} = \frac{5h^3}{12}\left[\frac{f_{n+7/12}^1 - f_n^1}{7h/12}\right] = \frac{5h^2}{7}\left[f_{n+7/12}^1 - f_n^1\right]$$

$$\left[\frac{5h^2/7\left[f_{n+1}-f_{n+7/12}\right]}{5h/12}\right]-\left[\frac{5h^2/7\left[f_{n+1}-f_n^1\right]}{h}\right]$$

$$=\frac{12h}{7}\left[f_{n+1}-f_{n+7/12}\right]-\frac{5h}{7}\left[f_{n+1}-f_n\right]$$

$$=\frac{h}{7}\left[12f_{n+1}-12f_{n+7/12}-5f_{n+1}+5f_n\right]$$

$$\frac{5h^3f_n^{11}}{12}=\frac{h}{7}\left[7f_{n+1}-12f_{n+7/12}+5f_n\right]\text{.............(6)}$$

Combining equations (6) and (4) above, we have:

$$\therefore y_{n+2}=y_{n+1}+\frac{h}{2}\left[3f_{n+1}-f_n\right]+\frac{h}{7}\left[7f_{n+1}-12f_{n+7/12}+5f_n\right]$$

$$y_{n+2}=y_{n+1}+\frac{h}{14}\left[21f_{n+1}-7f_n+14f_{n+1}-24f_{n+7/12}+10f_n\right]$$

$$\therefore y_{n+2}=y_{n+1}+\frac{h}{14}\left[35f_{n+1}+3f_n-24f_{n+7/12}\right]\text{...................(Scheme 3)}$$

3.0 Convergence Of The Methods

Numerical method is convergent if

$$\lim_{h\to 0}\max_{n=0,1\ldots x}\left\|e_n\right\|=0$$

To prove that a linear multistep method is convergent, it is sufficient to show that the method is consistent as well as zero-stable. [6]

Scheme 1

$$y_{n+2}=y_n+\frac{h}{3}[8f_{n+1}-4f_{n+1/2}+2f_n]$$

The error term is

$$y_{n+2}-y_n=\frac{h}{3}[8f_{n+1}-4f_{n+1/2}+2f_n]\text{..........(7)}$$

Consistency

From equation (7), the first characteristic polynomial $\rho(\xi)$ is given by

$$\rho(\xi)=\sum_{j=0}^{2}\alpha_j\xi^j$$

$$\rho(\xi)=\xi^2-1$$

$$\rho(1)=1-1=0$$

$$\rho^1(\xi)=2\xi$$

$$\rho^1(1)=2(1)-0(1)=2$$

The second characteristic polynomial $\sigma(\xi)$ is given by

$$\sigma(\xi)=\sum_{j=0}^{2}\alpha_j\xi^j$$

$$\sigma(\xi)=\frac{8}{3}-\frac{4}{3}-\frac{2}{3}=\frac{6}{3}=2$$

From equations above, we have

(i) $\rho(1)=0$

(ii) $\rho^1(1)=\sigma(1)$

Hence scheme 1 is consistent.

Zero-stability

The roots of the first characteristics polynomial are given by $\rho(\xi)=\xi^2-1=0$

i.e. $\xi=-1$ or $\xi=1$

Thus $\xi=-1,1$ which satisfy the zero-stability condition i.e $|\xi|\le 1$

We conclude that scheme 1 is convergent.

Scheme 2

$$y_{n+2}=y_n+\frac{h}{3}[7f_{n+1}-2f_n+f_{n-1}]$$

The error term is

$$y_{n+2}-y_n=\frac{h}{3}[7f_{n+1}-2f_n+f_{n-1}]\text{...........}(8)$$

Consistency

From equation (8), the first characteristic polynomial $\rho(\xi)$ is given by

$$\rho^1(\xi)=\sum_{j=0}^{2} j\alpha_j\xi^{j-1}$$

$$\rho(1)=\sum_{j=0}^{2}\alpha_j=1-1=0.$$

$$p^1(1)=\sum_{j-0}^{2} j\alpha_j=2(1)-0(1)=2$$

The second characteristic polynomial $\sigma(\xi)$ is given by

$$\sigma(\xi)=\sum_{j=0}^{2}\beta_j\xi^j$$

$$\sigma(1)=\sum_{j=0}^{2}\beta_j=\frac{7}{3}-\frac{2}{3}+\frac{1}{3}=\frac{1}{6}=2.$$

Thus, we have;

(i) $\rho(1)=0$

(ii) $\rho^1(1)=\sigma(1).$

Hence, scheme 2 is consistent

Zero-stability

$$\rho(\xi)=\xi^2-1=0$$

i.e. $\xi=-1$, $\xi=1$, which satisfy the zero stability condition.

Hence, since scheme 2 satisfied these conditions,

(i) $\rho(1)=0$

(ii) $\rho^1(1)=\sigma(1)$

(iii) Zero –stability condition

We conclude that scheme 2 is convergent.

Scheme 3

$$y_{n+2}=y_{n+1}+\frac{h}{14}\left[35f_{n+1}+3f_n-24f_{n+7/12}\right]$$

The Error term is

$$y_{n+2}-y_{n+1}=\frac{h}{14}\left[35f_{n+1}+3f_n-24f_{n+7/12}\right]\text{.............}(9)$$

Consistency:

From equation (9), the first characteristic polynomial $\rho(\xi)$ is given by

$$\rho(\xi) = \sum_{j=0}^{2} \alpha_j \xi^j$$

$$\rho(\xi) = \xi^2 - \xi$$

$$\rho(1) = 1 - 1 = 0$$

$$\rho^1(\xi) = 2\xi - 1$$

$$\rho^1(1) = 2(1) - 1 = 1$$

The second characteristic polynomial $\sigma(\xi)$ is given by

$$\sigma(\xi) = \sum_{j=0}^{1} \beta_j \xi^j$$

$$\sigma(1) = \frac{35}{14} + \frac{3}{14} - \frac{24}{14} = \frac{14}{14} = 1$$

we have

(i) $\rho(1) = 0$

(ii) $\rho^1(1) = \sigma(1)$

Hence since the scheme satisfies the above conditions, it is consistent

Zero-stability

The roots of the first characteristic polynomial $\rho(\xi)$ is given by

$$\rho(\xi) = \xi^2 - \xi = 0$$

$$\xi(\xi - 1) = 0$$

$$\xi = 0, \xi = 1$$

Since ξ satisfies the zero-stability conditions and it is also consistent, we conclude that scheme 3 is convergent.

4.0 Numerical Application and Comparison Of Results

We use the three (3) derived Methods to solve the following differential equation $y' = x + y; y(0) = 1, h = 0.1$. The results are obtained and compared for accuracy. The problem is implemented on computer using Microsoft Excel software/ package.

The results obtained from the three new schemes are compared with the exact solution and the old schemes.

Table 1:Scheme 1

PROBLEM: $y' = x + y; y(0) = 1, h = 0.1$

X	Exact Solution	$y_{n+2} = y_n + 2hf_{n+1}$	Error	$y_{n+2} = y_n + \frac{h}{3}[8f_{n+1} - 4f_{n+1/2} + 2f_n]$	Error
0.1	1.110341836	1.1103	4.1836E-05	1.11	3.41836E-04
0.2	1.242805516	1.24206	7.45516E-04	1.2426667	1.3889E-04
0.3	1.399717615	1.398712	1.005615E-03	1.39931111	4.06505E-04
0.4	1.583649395	1.5818024	1.846995E-03	1.583454078	1.95317E-04
0.5	1.797442541	1.7950724	2.370141E-03	1.796922717	5.19824E-04
0.6	2.044234601	2.0408168	3.420801E-03	2.04375417	4.83431E-04
0.7	2.327505415	2.3232357	4.26715E-03	2.326816163	6.89252E-04
0.8	2.651081857	2.6454639	5.6217957E-03	2.6503631714	7.18686E-04
0.9	3.019206222	3.0123284	6.877822E-03	3.01827824	9.26398E-04
1.0	3.436563657	3.4279295	8.634157E-03	3.4554494	1.8885743E-02

Table 2: Scheme 2

PROBLEM: $y' = x + y; y(0) = 1, h = 0.1$

X	Exact Solution	$y_{n+2} = y_n + 2hf_{n+1}$	Error	$y_{n+2}=y_n+\frac{h}{3}[7f_{n+1}-2f_n+f_{n-1}]$	Error
0.1	1.110341836	1.1103	4.1836E-05	1.1103	4.1836E-05
0.2	1.242805516	1.24206	7.45516E-04	1.2428	5.516E-06
0.3	1.399717615	1.398712	1.005615E-03	1.39960	1.17615E-04
0.4	1.583649395	1.5818024	1.846995E-03	1.58353	1.19395E-04
0.5	1.797442541	1.7950724	2.370141E-03	1.797210333	2.32221E-04
0.6	2.044234601	2.0408168	3.420801E-03	2.043963744	2.73857E-04
0.7	2.327505415	2.3232357	4.26715E-03	2.327105318	4.00097E-04
0.8	2.651081857	2.6454639	5.6217957E-03	2.650597746	4.84111E-04
0.9	3.019206222	3.0123284	6.877822E-03	3.018571088	6.35134E-04
1.0	3.436563657	3.4279295	8.634157E-03	3.435794647	7.6901E-04

Table 3: Scheme 3

PROBLEM: $y' = x + y; y(0) = 1, h = 0.1$

X	Exact Solution	$y_{n+2} = y_n + 2hf_{n+1}$	Error	$y_{n+2} = y_{n+1} + \frac{h}{14}\left[35f_{n+1} + 3f_n - 24f_{n+7/12}\right]$	Error
0.1	1.110341836	1.1103	4.1836E-05	1.11	1.0341836E-05
0.2	1.242805516	1.24206	7.45516E-04	1.2425	3.05516E-06
0.3	1.399717615	1.398712	1.005615E-03	1.399525001	1.97605E-04
0.4	1.583649395	1.5818024	1.846995E-03	1.583606251	4.3144E-04
0.5	1.797442541	1.7950724	2.370141E-03	1.797583815	1.412272E-04
0.6	2.044234601	2.0408168	3.420801E-03	2.044602694	3.65093E-04
0.7	2.327505415	2.3232357	4.26715E-03	2.32813993	6.34515E-04
0.8	2.651081857	2.6454639	5.6217957E-03	2.651245337	1.6348E-04
0.9	3.019206222	3.0123284	6.877822E-03	3.019554283	2.92063E-04
1.0	3.436563657	3.4279295	8.634157E-03	3.4357104808	5.41151E-04

5.0 Analysis of Results and Conclusion

From the Tables above, the new schemes are more accurate than the old schemes, as they produce less error [up to 4 decimal places] than the old schemes with error [up to 3 decimal places]. We can conclude that the three (3) new schemes are accurate as they produce results which are comparable to and even more accurate than those produced by other similar methods.

References

[1] Aikeu. R.C.: Stiff Computation. New York, Oxford: Oxford University press.(1985)

[2] Salisu, A.: High Orders Quasi- Runge- Kutta Methods by Refinement process for the solution of Initial Value Problems.M.Tech. dissertations, Federal University of Technology.Minna, Nigeria (2008), (Unpublished)

[3] Wright,W.M: Explicit General Linear Method With inherit Runge-kutta Stability. Numerical Algorithms 31: (2002),381-399.

[4] Adeboye, K.R. AND Ochoche, A.: A fifth order six-stage Runge-Kutta for solving initial value problems, Abacus Journal of the Mathematical Association of Nigeria, vol33 No 2A,(2006), P 85 – 98

[5] Scheid, F. :Numerical Analysis, Second Edition, Schaum's outlines, McGraw-Hill. USA, (1988), 187-188

[6] Butcher, J.C : The Numerical Analysis of Ordinary Differential Equations: Runge-Kutta and General linear method Wiley, Chichester.(1987),