# On the Model for Nuclear Safety Characterization from the Dynamical Structure of Reaction Diffusion Equation 

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#### Abstract

Quadratic Cost Function with Dynamic constraints can be solved by Penalty function Method but the convergence estimate may be poor or unavailable. However the Extended Conjugate Gradient Method (ECGM) will provide a numerical solution with an acceptable tolerance .This work considers the dynamical structure of reaction diffusion equation. Hill [1] outlined a general procedure for obtaining closed form representations of the solutions $u(x, t)$ and $v(x, t)$ for the linear reaction diffusion equation but the solution was not explicit. We now propose a control approach to transform coupled linear reaction diffusion equations to control problem. The resulting problem is admissible in Hilbert and Sobolev Spaces. Consequently, the model represents a Mathematical structure that is fundamental to Reaction diffusion problems such as chemical reaction equations, nuclear problems etc. The consequence leads to the derivation of the model for nuclear safety chacterization. The existence and uniqueness of solution is established for nuclear safety model.


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### 1.0 Introduction

Reaction Diffusion Systems are of great importance to Engineers, Scientists and Computational Mathematicians. Many researchers have worked on Coupled Linear Reaction Diffusion Equations but there are no explicit solutions. Hill [2] outlined a general procedure for obtaining a closed form representations of the solutions $u(x, t)$ and $v(x, t)$ for the linear reaction diffusion equation:.

$$
\begin{align*}
& \frac{\partial u}{\partial t}=D_{1} \nabla^{2} u-a u+b v \\
& \frac{\partial v}{\partial t}=D_{2} \nabla^{2} v+c u-d v \tag{1.1}
\end{align*}
$$

where $D_{1}, D_{2}, a, b, c$ and $d$ are all nonnegative constants.

[^0]He showed that closed form solutions of (1.1) can be given in terms of arbitrary integral heat functions $h_{1}(x, t)$ and $h_{2}(x, t)$.

That is functions that satisfy classical heat equation

$$
\begin{equation*}
\frac{\partial h}{\partial t}=\nabla h . \tag{1.2}
\end{equation*}
$$

and in particular; he established that the formal solutions of (1.1) are

$$
\begin{align*}
u(x, t)= & e^{-a t} h_{1}\left(x, D_{1} t\right) \\
& +\frac{b^{1 / 2} e^{-\lambda A t}}{\left(D_{1}-D_{2}\right)} \int_{D_{2} t}^{D_{1} t} e^{-\mu \xi}\left\{e^{1 / 2} \frac{\left(\xi-D_{2} t\right)^{1 / 2}}{D_{1} t-\xi} I_{1}(n) h_{1}(x, \xi)+b^{\frac{1}{2}} I_{0}(n) h_{2}(x, \xi)\right\} d \xi \\
v(x, t)= & e^{-a t} h_{2}\left(x, D_{2} t\right)+\frac{c^{1 / 2} e^{-\lambda A t}}{\left(D_{1}-D_{2}\right)} \int_{D_{2} t}^{D_{1} t} e^{-\mu \xi}\left\{e^{1 / 2} \frac{\left(D_{1} t-\xi\right)^{1 / 2}}{\xi-D_{2} t} I_{2}(\tau) h_{2}(x, \xi)+c^{\frac{1}{2}} I_{0}(\tau) h_{1}(x, \xi)\right\} d \xi,
\end{align*}
$$

where the constants $\lambda$ and $\mu$ are given by the following equations:

$$
\begin{align*}
& \lambda=\frac{\left(a D_{2}-d D_{1}\right)}{D_{1}-D_{2}}  \tag{1.5}\\
& \mu=\frac{(a-d)}{D_{1}-D_{2}} \tag{1.6}
\end{align*}
$$

$I_{0}, I_{1}, I_{2}$ are the usual modified Bessel functions and $\tau$ is given by

$$
\begin{equation*}
\tau=\frac{2(b c)^{1 / 2}}{D_{1}-D_{2}}\left[\left(D_{1} t-\xi\right)\left(\xi-D_{2} t\right)\right]^{\frac{1}{2}} \tag{1.7}
\end{equation*}
$$

Hill [2] considered the application of these general formulae to the stability problems arising from a model of an arms race which incorporates the features of deteriorating armaments.

This situation is as follows:
Richardson as reported in [5] proposed that the military spending of two nations locked in an arms race can be modelled by the following linear systems:

$$
\begin{align*}
& \frac{d p}{d t}(t)=-a p(t)+b q(t)+g  \tag{1.8}\\
& \frac{d q}{d t}(t)=c p(t)-d q(t)+h \tag{1.9}
\end{align*}
$$

Where $p(t)$ and $q(t)$ denote armament levels of the two nations at time $t$ and $a, b, c, d, g$ and $h$ denote positive constants. The constants $b$ and $c$ are called "Threat Coefficients" and they signify the degree to which a nation is stimulated by another nation's weapon stock to increase her own stocks. The constants $a$ and d, called "Fatigue Coefficients", are measures of prevailing economic circumstances which inhibit armament build-up. The constants $g$ and $h$ denote measures of the circumstances which prevent a complete disarmament in the situation when both nations have zero armaments. A "Balance of Power" situation results when the armament levels remain constant over a long period of time and these levels are given [ 2 ] by the following equation

$$
\begin{align*}
p_{0}= & \frac{g d+h b}{(a d-b c)}  \tag{1.10}\\
q_{0}= & \frac{g c+h a}{(a d-b c)}  \tag{1.11}\\
& \quad(a d-b c)>0
\end{align*}
$$

Gopalsamy as reported in [1] developed Richardson model and proposed that the armament levels $p(x, t)$ and $q(x, t)$ satisfy

$$
\begin{equation*}
\frac{\partial p}{\partial t}+e_{1} \frac{\partial p}{\partial x}=\frac{a_{1}^{2}}{2} \frac{\partial^{2} p}{\partial x^{2}}-a p+b q+g \tag{1.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial q}{\partial t}+e_{2} \frac{\partial p}{\partial x}=\frac{a_{2}^{2}}{2} \frac{\partial^{2} p}{\partial x^{2}}+c q-d p+h \tag{1.13}
\end{equation*}
$$

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where $e_{1}, e_{2}, a_{1}$ and $a_{2}$ denote positive constants that are also previously defined.
Hill [2] further developed a model and asserted that in order to investigate the stability of power situation ( $p_{0}, q_{0}$ ) given by (1.10) and (1.11) he sets

$$
\begin{align*}
& p(x, t)=p_{0}+u(x, t)  \tag{1.14}\\
& \text { and } \\
& q(x, t)=q_{0}+v(x, t) \tag{1.15}
\end{align*}
$$

so that from (1.12) and (1.13), we have that:

$$
\begin{align*}
\frac{\partial u}{\partial t} & =D_{1} \frac{\partial^{2} u}{\partial x^{2}}-e_{1} \frac{\partial u}{\partial x}+a u-b v \\
\frac{\partial v}{\partial t} & =D_{2} \frac{\partial^{2} v}{\partial x^{2}}-e_{2} \frac{\partial v}{\partial x}+c u-d v \\
D_{i} & =\frac{\alpha_{i}^{2}}{2}(i=1,2)  \tag{1.16}\\
u(x, 0) & =0, v(x, 0)=0, u(0, t)=u_{0}, v(0, t)=v_{0} \\
u(x, t) & , v(x, t) \rightarrow 0 \text { as } x \rightarrow \infty .
\end{align*}
$$

### 2.0 Methodology

Despite the large amount of qualitative theory for reaction diffusion equations that have been carried out, the application and interpretation of the results are difficult.

Because of the above short comings, we are concerned with the transformation of linear reaction diffusion equations into a control problem so that the application of optimization techniques can be employed for the solution of the resulting problem. Basically, this work considers the transformation of Coupled Linear Reaction Diffusion Equations into optimization so that some numerical methods can be used for the solution of resulting problem. This will make possible the utilization of known and unknown features of reaction diffusion equations to be used to advance theorems and propositions in Mathematical Sciences and Computation.

We shall now state and prove an important theorem in this work.

## 1. Theorem:

The reaction diffusion problem (1.16) is equivalent to the following control problem:

$$
\text { Minimize } \int_{0}^{T}\left\{a u^{2}(t)+a v^{2}(t)\right\} d t
$$

Subject to

$$
\dot{u}(t)-\dot{v}(t)=c v(t)+d u(t)
$$

## Proof:

We shall now transform the whole reaction diffusion system (1.16) into a control problem in the following manner.
Minimize $\int_{0}^{1} \int_{0}^{1}\left[u^{2}(x, t)+v^{2}(x, t)\right] d x d t$
Subject to

$$
\begin{align*}
& \frac{\partial u}{\partial t}-D_{1} \frac{\partial^{2} u}{\partial x^{2}}+e_{1} \frac{\partial u}{\partial x}-a u+b v=0 \\
& \frac{\partial v}{\partial t}-D_{2} \frac{\partial^{2} v}{\partial x^{2}}+e_{2} \frac{\partial v}{\partial x}-c u+d v=0 \tag{2.1}
\end{align*}
$$

$0 \leq x, t \leq 1$,
with the following initial and boundary conditions:
$u(x, 0)=0, v(x, 0)=0, u(0, t)=u_{0}, v(0, t)=v_{0}$
$u(x, t), v(x, t) \rightarrow 0$ as $x \rightarrow \infty$.
$\frac{\partial u(x, 1)}{\partial t}=\frac{\partial v(x, 1)}{\partial t}=1$.
The boundary conditions at $x$ equals zero represent the fact that both nations locked in the arms race are maintaining a perfect level of undeteriorated strategic weapon system and the integral given by (2.1) is a measure of military spending.
To obtain an explicit solution of the boundary value problem (1.16), Gopalsamy as reported in [2] assumed that $e_{1}=$ $e_{2}, D_{1}=D_{2}$ and $a=d$. We also adopt these values for simplicity and consistency.

Let $v(x, t)=\sum_{i=1}^{\infty} \dot{v}_{i}(t) \frac{\operatorname{sin\pi ix}}{l} .0 \leq x \leq l \equiv 1$,

$$
u(x, t)=\sum_{i=1}^{\infty} \dot{u}_{i}(t) \frac{\sin \pi i x}{l} .0 \leq x \leq l \equiv 1
$$

Thus we have

$$
\begin{aligned}
& v_{i}(x, t)=\sum_{i=1}^{\infty} v_{i}(t) \sin \pi i x \\
& u_{i}(x, t)=\sum_{i=1}^{\infty} u_{i}(t) \sin \pi i x \\
& v_{x x}(x, t)=-\pi^{2} i^{2} \sum_{i=1}^{\infty} v_{i}(t) \sin \pi i x . \\
& u_{x x}(x, t)=-\pi^{2} i^{2} \sum_{i=1}^{\infty} u_{i}(t) \sin \pi i x \\
& v^{2}(x, t)=\sum_{i=1}^{\infty} v_{i}^{2}(t) \sin ^{2} \pi i x . \\
& u^{2}(x, t)=\sum_{i=1}^{\infty} u_{i}^{2}(t) \sin ^{2} \pi i x .
\end{aligned}
$$

And on substituting the values of $v^{2}(x, t), u^{2}(x, t)$ in the integral (2.1), we obtain:

$$
\begin{align*}
& \int_{0}^{1} \int_{0}^{1}\left[v^{2}(x, t)+u^{2}(x, t)\right] d x d t= \\
& \int_{0}^{1} \int_{0}^{1}\left[\sum_{i=1}^{\infty} v_{i}^{2}(t) \sin ^{2} \pi i x+\sum_{i=1}^{\infty} u_{i}^{2}(t) \sin ^{2} \pi i x\right] d x d t . \\
& =\frac{1}{2} \int_{0}^{1} \int_{0}^{1}\left[\sum_{i=1}^{\infty} v_{i}^{2}(t)[1-\cos 2 \pi i x]+\sum_{i=1}^{\infty} u_{i}^{2}(t)[1-\cos 2 \pi i x]\right] d x d t .  \tag{2.3}\\
& =\frac{1}{2} \int_{0}^{1}\left\{\sum_{i=1}^{\infty} v_{i}^{2}(t)\left[x-\frac{\sin \pi i x}{2 \pi i}\right]_{0}^{1}+\sum_{i=1}^{\infty} u_{i}^{2}(t)\left[x-\frac{\sin \pi i x}{2 \pi i}\right]_{0}^{1}\right\} d t . \\
& =\frac{1}{2} \int_{0}^{1}\left\{\sum_{i=1}^{\infty} v_{i}^{2}(t)[1-0-(+0-0)]+\sum_{i=1}^{\infty} u_{i}^{2}(t)[1-0-(+0-0)]\right\} d t . \\
& =\frac{1}{2} \int_{0}^{1}\left\{\sum_{i=1}^{\infty} v_{i}^{2}(t)+\sum_{i=1}^{\infty} u_{i}^{2}(t)\right\} d t .
\end{align*}
$$

Since it is a minimization problem the $\frac{1}{2}$ in the right hand side can be omitted.
Thus we have

$$
\begin{aligned}
& \int_{0}^{1} \int_{0}^{1}\left[v^{2}(x, t)+u^{2}(x, t)\right] d x d t \\
& =\int_{0}^{1}\left\{\sum_{i=1}^{\infty} v_{i}^{2}(t)+\sum_{i=1}^{\infty} u_{i}^{2}(t)\right\} d t
\end{aligned}
$$

Following the idea of Gopalsamy as reported in [1]

$$
\begin{equation*}
e_{1}=e_{2}=1 \tag{2.4}
\end{equation*}
$$

Equating the constraints in (2.1), we obtain

$$
\frac{\partial u}{\partial t}-D_{1} \frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial u}{\partial x}+a u-b v=\frac{\partial v}{\partial t}-D_{2} \frac{\partial^{2} v}{\partial x^{2}}+\frac{\partial v}{\partial x}-c u+d v
$$

On substituting values for $u_{i}, v_{i}, u_{x x}, v_{x x}, u, v$ in the last equation, we obtain

$$
\begin{aligned}
\sum_{i=1}^{\infty} \dot{u}_{i}(t) \sin \pi i x & -D_{1} \pi^{2} i^{2} \sum_{i=1}^{\infty} \sin \pi i x+a \sum_{i=1}^{\infty} u_{i}(t) \sin \pi i x-b \sum_{i=1}^{\infty} v_{i}(t) \sin \pi i x \\
& =\sum_{i=1}^{\infty} \dot{v}_{i}(t) \sin \pi i x-D_{2} \pi^{2} i^{2} \sum_{i=1}^{\infty} \sin \pi i x-c \sum_{i=1}^{\infty} u_{i}(t) \sin \pi i x+d \sum_{i=1}^{\infty} v_{i}(t) \sin \pi i x
\end{aligned}
$$

Dividing both sides by $\sin \pi i x$, since $\sin \pi i x \neq 0, \forall 0<x<1$ we obtain

$$
\begin{aligned}
& \sum_{i=1}^{\infty} \dot{u}_{i}(t)+D_{1} \pi_{i}^{2} \sum_{i=1}^{\infty} \mathrm{i}^{2}+a \sum_{i=1}^{\infty} u_{i}(t)-b \sum_{i=1}^{\infty} v_{i}(t) \\
& =\sum_{i=1}^{\infty} \dot{v}_{i}(t)+D_{2}^{2} \pi_{i}^{2} \sum_{i=1}^{\infty} \mathrm{i}^{2}-c \sum_{i=1}^{\infty} u_{i}(t)+d \sum_{i=1}^{\infty} v_{i}(t) .
\end{aligned}
$$

Rearranging and dropping the summation sign, we obtain

$$
\begin{align*}
& \qquad \begin{array}{l}
\dot{u}_{1}(t)-\dot{v}_{1}(t)=\left[D_{2} \pi^{2} 1^{2}+d+b\right] v_{1}(t)+\left[-D_{1} \pi^{2} 1^{2}-a-c\right] u_{1}(t) \\
\quad+\dot{u}_{2}(t)-\dot{v}_{2}(t)=\left[D_{2} \pi^{2} 2^{2}+d+b\right] v_{2}(t)+\left[-D_{1} \pi^{2} 2^{2}-a-c\right] u_{2}(t)
\end{array} \\
& \quad+\ldots+ \\
& \dot{u}_{n}(t)-\ldots v_{n}(t)=\left[D_{2} \pi^{2} n^{2}+d+b\right] v_{n}(t)+\left[-D_{1} \pi^{2} n^{2}-a-c\right] u_{n}(t) . \\
& \text { The above can be put in a compact form in the following manner: }  \tag{2.5}\\
& \\
& \text { Where } \quad \begin{array}{l}
u_{i}(t)-\dot{v}_{i}(t)=C v_{i}(t)+D u_{i}(t) .
\end{array}  \tag{2.6}\\
& \quad \begin{array}{l}
C=D_{2} \pi^{2} i^{2}+d+b, \\
D=-D_{1} \pi^{2} i^{2}-a-c, \quad i=1,2, \ldots, n .
\end{array} \quad i=1,2, \ldots, n .
\end{align*}
$$

Consequently, problem (2.1) reduces to
Minimize $\quad \int_{0}^{T}\left\{\sum_{i=1}^{\infty} v_{i}^{2}(t)+\sum_{i=1}^{\infty} u_{i}^{2}(t)\right\} d t$
Subject to

$$
\begin{equation*}
\dot{u}_{i}(t)-\dot{v}_{i}(t)=C v_{i}(t)+D u_{i}(t) \tag{2.8}
\end{equation*}
$$

Where

$$
\begin{array}{lr}
C=D_{2} \pi^{2} i^{2}+d+b, & i=1,2, \ldots, n . \\
D=-D_{1} \pi^{2} i^{2}-a-c, & i=1,2, \ldots, n .
\end{array}
$$

This can be put as a one dimensional problem in the form:

$$
\text { Minimize } \int_{0}^{T}\left\{a u^{2}(t)+b v^{2}(t)\right\} d t
$$

Subject to

$$
\begin{equation*}
\dot{u}(t)-\dot{v}(t)=c v(t)+d u(t) \tag{2.9}
\end{equation*}
$$

This completes the proof of the theorem.
This problem with dynamic constraint has enabled us to model nuclear safety as a cost functional with dynamic constraint which is reported in the next section

### 3.0 Nuclear Safety Characterization

The benefits derived from nuclear technology are of great importance. However, the attendance accident gives concern to mankind. To date, there have been five serious accidents reported in the world since 1970 (one at Three Mile Island in 1979; one at Chernobyl in 1986; and three at Fukushima-Daiichi in 2011) [3, 4]. This suggests the need for finding lasting solution, through mathematical approach to serious accident happening in the nuclear world with respect to the rate of heat that usually causes the release of radiation after damaging the containment structure. In our earlier work on nuclear safety [14], we used part of the formalism of our model in section 2 to structure nuclear tokens (from the rate of heat equations of nuclear reactors) in the form of quadratic functional model, which is solvable by the Extended Conjugate Gradient Method (ECGM) Algorithm formulated by Ibiejugba et al [5] and others [6-10]. The construction of operator $\widetilde{\boldsymbol{E}}$ is proposed and the existence and uniqueness of solution to our resulting continuous quadratic cost functional will be established in next section.

### 4.0 Construction of Operator $\widetilde{\boldsymbol{E}}$

Consider the quadratic cost functional equation of the form

$$
\begin{equation*}
\operatorname{Min} J(v, u, w)=\int_{0}^{T}\left\{a v^{2}(t)+b u^{2}(t)+c w^{2}(t)\right\} d t \tag{4.1}
\end{equation*}
$$

Subject to

$$
\begin{equation*}
\dot{v}-\dot{u}+\dot{w}-d w=0 \tag{4.2}
\end{equation*}
$$

Equations (4.1)-(4.2) is transformed to quadratic continuous cost functional with the introduction of a penalty constant $\mu$ in the following form:

$$
\begin{equation*}
\operatorname{Min} J(v(t), u(t), w(t), \mu)=\int_{0}^{T}\left\{a v^{2}(t)+b u^{2}(t)+c w^{2}(t)+\mu\|\dot{v}(\mathrm{t})-\dot{u}(t)+\dot{w}(\mathrm{t})-d w(t)\|^{2}\right\} d t \tag{4.3}
\end{equation*}
$$

It has been shown [8] that it is self-adjoint and Hermittian. It is now left to prove that (4.1)-(4.2) is bounded. To do that we proceed as follows:
Suppressing $t$ and expanding (4.3), we have

$$
\begin{gather*}
\operatorname{Min} J(v, u, w, \mu)=\int_{0}^{T}\left\{a v^{2}+b u^{2}+c w^{2}+\mu(\dot{v}-\dot{u}+\dot{w}-d w)^{T}(\dot{v}-\dot{u}+\dot{w}-d w)\right\} d t \\
=\int_{0}^{T}\left\{a v^{2}+b u^{2}+c w^{2}+\mu \dot{v}^{2}+\mu \dot{u}^{2}+\mu \dot{w}^{2}+\mu(d w)^{2}-2 \mu \dot{v} \dot{u}+2 \mu \dot{v} \dot{w}-2 \mu \dot{v} d w-2 \mu \dot{u} \dot{w}+2 \mu \dot{u} d w\right. \\
-2 \mu \dot{w} d w\} d t \tag{4.4}
\end{gather*}
$$

We now construct an operator $\tilde{E}_{i j}$ such that equation (4.4) can be written as follows:

$$
\begin{equation*}
J_{\mu}(v, u, w)=\int_{0}^{T} Z \widetilde{\boldsymbol{E}} Z d t \tag{4.5}
\end{equation*}
$$

Where

$$
\tilde{E}=\left(\begin{array}{llllll}
E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16}  \tag{4.6}\\
E_{21} & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\
E_{31} & E_{32} & E_{33} & E_{34} & E_{35} & E_{36} \\
E_{41} & E_{42} & E_{43} & E_{44} & E_{45} & E_{46} \\
E_{51} & E_{52} & E_{53} & E_{54} & E_{55} & E_{56} \\
E_{61} & E_{62} & E_{63} & E_{64} & E_{65} & E_{66}
\end{array}\right)
$$

And

$$
Z=\left(\begin{array}{c}
\dot{v}  \tag{4.7}\\
v \\
\dot{u} \\
u \\
\dot{w} \\
W
\end{array}\right)^{T}
$$

It is this control operator $\tilde{E}$ that we seek to determine. Equation (4.5) can be written in the following equivalent form:

$$
\begin{align*}
& J_{\mu}(v, u, w)=\int_{0}^{T}\left\{\left(\begin{array}{c}
\dot{v} \\
v \\
\dot{u} \\
u \\
\dot{w} \\
w
\end{array}\right)^{T}\left(\begin{array}{llllll}
E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\
E_{21} & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\
E_{31} & E_{32} & E_{33} & E_{34} & E_{35} & E_{36} \\
E_{41} & E_{42} & E_{43} & E_{44} & E_{45} & E_{46} \\
E_{51} & E_{52} & E_{53} & E_{54} & E_{55} & E_{56} \\
E_{61} & E_{62} & E_{63} & E_{64} & E_{65} & E_{66}
\end{array}\right)\left(\begin{array}{c}
\dot{v} \\
v \\
\dot{u} \\
u \\
\dot{w} \\
w
\end{array}\right)\right\} d t  \tag{4.8}\\
& =\int_{0}^{T}\left\{\left(\begin{array}{l}
E_{11} \dot{v}+E_{21} v+E_{31} \dot{u}+E_{41} u+E_{51} \dot{w}+E_{61} w \\
E_{12} \dot{v}+E_{22} v+E_{32} \dot{u}+E_{42} u+E_{52} \dot{w}+E_{62} w \\
E_{13} \dot{v}+E_{23} v+E_{33} \dot{u}+E_{43} u+E_{53} \dot{w}+E_{63} w \\
E_{14} \dot{v}+E_{24} v+E_{34} \dot{u}+E_{44} u+E_{54} \dot{w}+E_{64} w \\
E_{15} \dot{v}+E_{25} v+E_{35} \dot{u}+E_{45} u+E_{55} \dot{w}+E_{65} w \\
E_{16} \dot{v}+E_{26} v+E_{36} \dot{u}+E_{46} u+E_{56} \dot{w}+E_{66} w
\end{array}\right)\left(\begin{array}{c}
v \\
\dot{u} \\
u \\
\dot{w} \\
w
\end{array}\right)\right\} d t \tag{4.9}
\end{align*}
$$

$$
\begin{aligned}
J_{\mu}(v, u, w)=\int_{0}^{T}\{ & E_{11} \dot{v}^{2}+E_{22} v^{2}+E_{33} \dot{u}^{2}+E_{44} u^{2}+E_{55} \dot{w}^{2}+E_{66} w^{2}+\left(E_{12}+E_{21}\right) \dot{v} v+\left(E_{13}+E_{31}\right) \dot{v} \dot{u} \\
& +\left(E_{14}+E_{41}\right) \dot{v} u+\left(E_{15}+E_{51}\right) \dot{v} w+\left(E_{16}+E_{61}\right) \dot{v} w+\left(E_{23}+E_{32}\right) \dot{u} v+\left(E_{24}+E_{42}\right) u v \\
& +\left(E_{25}+E_{52}\right) \dot{w} v+\left(E_{26}+E_{62}\right) v w+\left(E_{34}+E_{43}\right) \dot{u} u \\
& \left.+\left(E_{35}+E_{53}\right) \dot{u} w+\left(E_{36}+E_{63}\right) \dot{u} w+\left(E_{45}+E_{54}\right) \dot{w} \dot{u}+\left(E_{46}+E_{64}\right) u w+\left(E_{56}+E_{65}\right) \dot{w} w\right\} d t(4.10)
\end{aligned}
$$

Comparing coefficients in (4.4) and (4.10) and simplifying, we obtain the following:

$$
\begin{gathered}
E_{11}=\mu, E_{12}=0, E_{13}=-\mu, E_{14}=0, E_{15}=\mu, E_{16}=-\mu d \\
E_{21}=0, E_{22}=a, E_{23}=0, E_{24}=0, E_{25}=0, E_{26}=0, \\
E_{31}=-\mu, E_{32}=0, E_{33}=\mu, E_{34}=0, E_{35}=-\mu, E_{36}=\mu d, \\
E_{41}=0, E_{42}=0, E_{43}=0, E_{44}=b, E_{45}=0, E_{46}=0, \\
E_{51}=\mu, E_{52}=0, E_{53}=-\mu, E_{54}=0, E_{55}=\mu, E_{56}=-\mu d, \\
E_{61}=-\mu d, E_{62}=0, E_{63}=\mu d, E_{64}=0, E_{65}=-\mu d \text { and } E_{66}=c+\mu d^{2}
\end{gathered}
$$

The $E_{i j} \mathrm{~s}^{\prime}$ above are the elements of the control operator $\tilde{E}$ which can now be written in form of matrix as

$$
\tilde{E}=\left(\begin{array}{cccccc}
\mu & 0 & -\mu & \mu & \mu & -\mu d  \tag{4.11}\\
0 & a & 0 & 0 & 0 & 0 \\
-\mu & 0 & \mu & 0 & -\mu & \mu d \\
0 & 0 & 0 & b & 0 & 0 \\
\mu & 0 & -\mu & 0 & \mu & -\mu d \\
-\mu d & 0 & \mu d & 0 & -\mu d & c+\mu d^{2}
\end{array}\right)
$$

Therefore, (4.11) is the required control operator $\tilde{E}$. On factorizing, it yields

$$
\tilde{E}=\mu\left(\begin{array}{cccccc}
1 & 0 & -1 & 0 & 1 & -d  \tag{4.12}\\
0 & \mu^{-1} a & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & -1 & d \\
0 & 0 & 0 & \mu^{-1} b & 0 & 0 \\
1 & 0 & -1 & 0 & 1 & -d \\
-d & 0 & d & 0 & -d & \mu^{-1} c+d^{2}
\end{array}\right)
$$

Let us take the limit of $\tilde{E}$, that is

$$
\begin{align*}
& \lim _{\mu \rightarrow \infty} \tilde{E}=\lim _{\mu \rightarrow \infty}\left\{\left(\begin{array}{cccccc}
1 & 0 & -1 & 0 & 1 & -d \\
0 & \mu^{-1} a & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & -1 & d \\
0 & 0 & 0 & \mu^{-1} b & 0 & 0 \\
1 & 0 & -1 & 0 & 1 & -d \\
-d & 0 & d & 0 & -d & \mu^{-1} c+d^{2}
\end{array}\right)\right\} \\
& =\lim _{\mu \rightarrow \infty} \mu \times \lim _{\mu \rightarrow \infty}\left(\begin{array}{ccccccc}
1 & 0 & -1 & 0 & 1 & -d \\
0 & \mu^{-1} a & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & -1 & d \\
0 & 0 & 0 & \mu^{-1} b & 0 & 0 \\
1 & 0 & -1 & 0 & 1 & -d \\
-d & 0 & d & 0 & -d & \mu^{-1} c+d^{2}
\end{array}\right)  \tag{4.13}\\
& =\lim _{\mu \rightarrow \infty} \mu \times\left(\begin{array}{cccccc}
1 & 0 & -1 & 0 & 1 & -d \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & -1 & d \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 1 & -d \\
-d & 0 & d & 0 & -d & d^{2}
\end{array}\right) \tag{4.14}
\end{align*}
$$

Hence,

$$
\left.\lim _{\mu \rightarrow \infty}|\tilde{E}|=\lim _{\mu \rightarrow \infty} \mu \times \left\lvert\, \begin{array}{cccccc}
1 & 0 & -1 & 0 & 1 & -d  \tag{4.15}\\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & -1 & d \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 1 & -d \\
-d & 0 & d & 0 & -d & d^{2}
\end{array}\right.\right) \left\lvert\, \begin{aligned}
& =\infty \times 0=0
\end{aligned}\right.
$$

This implies,

$$
\begin{equation*}
\lim _{\mu \rightarrow \infty} J_{\mu}(v, u, w)=0 \tag{4.16}
\end{equation*}
$$

Hence our cost functional is bounded, Hermittian and self-adjoint, therefore solution exists and it will be superlinearly convergent if an appropriate penalty function method is applied.

### 5.0 Conclusion

The consequences of the model in section 2 have allowed us to formulate model for nuclear safety in dynamical form as done in section 3 and 4. Also we are able to prove the Existence and Uniqueness of solution to our Nuclear safety characterization model. This will allow us to explore the Extended Conjugate Gradient Method Algorithm (ECGM) [5-10] for the solution of our control problem. This result is very significant because we can transform many modelling problems on reaction diffusion system into dynamical structure and solved numerically. More work will be done in the numerical computation aspect of the work.

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