Journal of the Nigerian Association of Mathematical Physics
***Volume* 22 (November, 2012), pp 407 – 410**

A Control Approach to the Existence and Uniqueness of Solution of Nuclear Reactor Safety Characterization

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Abstract

It has been proved that the cheapest means of power generation is through nuclear technology. This system devoice of emitting carbon into the atmosphere thereby causing environmental pollution that is dangerous to the human health. Despite the advantages associated with nuclear technology in power generation, its attendance accidents can be catastrophic to mankind. In this work, a continuous quadratic cost functional model for nuclear reactor safety was developed based on the residual equation of parametric heat equations and a control operator $\tilde{E}$ is established to give an alternative proof for the existence and uniqueness of the solution to our control problem. It is proved that the control problem is bounded, self adjoint and Hermittian. Consequently the control problem has an optimal solution and it is unique. A version of Conjugate Gradient (CGM) Algorithm can be formulated to solve our resulting control problem.

CR: 90C20
Keywords: Penalty parameter, Continuous, Cost Functional, Operator, Characterization.

1.0 Introduction

The benefits derived from nuclear technology are of great importance. However, the attendance accident gives concern to mankind. To date, there have been five serious accidents reported in the world since 1970 (one at Three Mile Island in 1979; one at Chernobyl in 1986; and three at Fukushima-Daiichi in 2011)[1, 2]. This leads us to finding lasting solution, through mathematical approach to serious accident happening in the nuclear world with respect to the rate of heat that usually causes the release of radiation after damaging the containment structure. In our earlier work on nuclear safety [3], we structured nuclear tokens (from the rate of heat equations of nuclear reactors) in the form of a quadratic functional model, which was solved by the Conjugate Gradient Method (CGM) Algorithm [4 - 10]. In this work, based on the construction of operator $\tilde{\boldsymbol{E}}$ a method is proposed to prove the existence and uniqueness of solution to our resulting continuous quadratic cost functional.

2.0 Main Result: Construction of Operator $\tilde{\boldsymbol{E}}$

Consider the quadratic cost functional equation of the form

$$Min\ J(v,u,w) = \int_0^T \{av^2(t) + bu^2(t) + cw^2(t)\}dt \qquad (1)$$

Subject to

$$\dot{v} - \dot{u} + \dot{w} - dw = 0 \qquad (2)$$

Equations (1)-(2) can be transform to quadratic continuous cost functional with the introduction of a penalty constant μ in the following form:

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$$Min\ \ J(v(t),u(t),w(t),\mu)=\int_0^T\{av^2(t)+bu^2(t)+cw^2(t)+\mu\|\dot{v}(t)-\dot{u}(t)+\dot{w}(t)-dw(t)\|^2\}dt \quad (3)$$

It has been shown [6] that it is self-adjoint and Hermittian. It is now left to prove that (1)-(2) is bounded. To do that we proceed as follows:
Suppressing t and expanding (3), we have

$$Min\ \ J(v,u,w,\mu)=\int_0^T\{av^2+bu^2+cw^2+\mu(\dot{v}-\dot{u}+\dot{w}-dw)^T(\dot{v}-\dot{u}+\dot{w}-dw)\}dt$$

$$=\int_0^T\{av^2+bu^2+cw^2+\mu\dot{v}^2+\mu\dot{u}^2+\mu\dot{w}^2+\mu(dw)^2-2\mu\dot{v}\dot{u}+2\mu\dot{v}\dot{w}-2\mu\dot{v}dw-2\mu\dot{u}\dot{w}+2\mu\dot{u}dw - 2\mu\dot{w}dw\}dt \quad (4)$$

We now construct an operator $\tilde{E}_{ij}$ such that equation (4) can be written as follows:

$$J_\mu(v,u,w)=\int_0^T Z\,\tilde{\boldsymbol{E}}Zdt \quad (5)$$

Where

$$\tilde{E}=\begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16}\\ E_{21} & E_{22} & E_{23} & E_{24} & E_{25} & E_{26}\\ E_{31} & E_{32} & E_{33} & E_{34} & E_{35} & E_{36}\\ E_{41} & E_{42} & E_{43} & E_{44} & E_{45} & E_{46}\\ E_{51} & E_{52} & E_{53} & E_{54} & E_{55} & E_{56}\\ E_{51} & E_{62} & E_{63} & E_{64} & E_{65} & E_{66}\end{pmatrix} \quad (6)$$

And

$$Z=\begin{pmatrix}\dot{v}\\ v\\ \dot{u}\\ u\\ \dot{w}\\ w\end{pmatrix}^T \quad (7)$$

It is this control operator $\tilde{E}$ that we seek to determine. Equation (6) can be written in the following equivalent form:

$$J_\mu(v,u,w)=\int_0^T\left\{\begin{pmatrix}\dot{v}\\ v\\ \dot{u}\\ u\\ \dot{w}\\ w\end{pmatrix}^T\begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16}\\ E_{21} & E_{22} & E_{23} & E_{24} & E_{25} & E_{26}\\ E_{31} & E_{32} & E_{33} & E_{34} & E_{35} & E_{36}\\ E_{41} & E_{42} & E_{43} & E_{44} & E_{45} & E_{46}\\ E_{51} & E_{52} & E_{53} & E_{54} & E_{55} & E_{56}\\ E_{61} & E_{62} & E_{63} & E_{64} & E_{65} & E_{66}\end{pmatrix}\begin{pmatrix}\dot{v}\\ v\\ \dot{u}\\ u\\ \dot{w}\\ w\end{pmatrix}\right\}dt \quad (8)$$

$$=\int_0^T\left\{\begin{pmatrix} E_{11}\,\dot{v}+E_{21}\,v+\ E_{31}\,\dot{u}+E_{41}\,u+\ E_{51}\,\dot{w}+E_{61}\,w\\ E_{12}\,\dot{v}+E_{22}\,v+\ E_{32}\,\dot{u}+E_{42}\,u+\ E_{52}\,\dot{w}+E_{62}\,w\\ E_{13}\,\dot{v}+E_{23}\,v+\ E_{33}\,\dot{u}+E_{43}\,u+\ E_{53}\,\dot{w}+E_{63}\,w\\ E_{14}\,\dot{v}+E_{24}\,v+\ E_{34}\,\dot{u}+E_{44}\,u+\ E_{54}\,\dot{w}+E_{64}\,w\\ E_{15}\,\dot{v}+E_{25}\,v+\ E_{35}\,\dot{u}+E_{45}\,u+\ E_{55}\,\dot{w}+E_{65}\,w\\ E_{16}\,\dot{v}+E_{26}\,v+\ E_{36}\,\dot{u}+E_{46}\,u+\ E_{56}\,\dot{w}+E_{66}\,w\end{pmatrix}^T\begin{pmatrix}\dot{v}\\ v\\ \dot{u}\\ u\\ \dot{w}\\ w\end{pmatrix}\right\}dt \quad (9)$$

Simplifying (9), we obtain

$$J_\mu(v,u,w)=\int_0^T\{E_{11}\,\dot{v}^2+E_{22}\,v^2+\ E_{33}\,\dot{u}^2+E_{44}\,u^2+\ E_{55}\,\dot{w}^2+E_{66}\,w^2+(E_{12}\ +E_{21})\dot{v}v+(E_{13}\ +E_{31})\dot{v}\dot{u}$$
$$+(E_{14}\ +E_{41})\dot{v}u\quad+(E_{15}+E_{51})\dot{v}\dot{w}\ +(E_{16}\ +E_{61})\dot{v}w\ +(E_{23}\ +E_{32})\dot{u}v+(E_{24}\ +E_{42})uv$$
$$+(E_{25}\ +E_{52})\dot{w}v+(E_{26}\ +E_{62})vw+(E_{34}\ +E_{43})\dot{u}u$$
$$+(E_{35}\ +E_{53})\dot{u}\dot{w}+(E_{36}\ +E_{63})\dot{u}w+(E_{45}\ +E_{54})\dot{w}u+(E_{46}\ +E_{64})uw+(E_{56}\ +E_{65})\dot{w}w\}dt \quad (10)$$

Comparing coefficients in (4) and (10) and simplifying, we obtain the following:

$E_{11}=\mu,\ E_{12}=0,\ E_{13}=-\mu,\ E_{14}=0,\ E_{15}=\mu,\ E_{16}=-\mu d$
$E_{21}=0,\ E_{22}=a,\ E_{23}=0,\ E_{24}=0,\ E_{25}=0,\ E_{26}=0,$
$E_{31}=-\mu,\ E_{32}=0,\ E_{33}=\mu,\ E_{34}=0,\ E_{35}=-\mu,\ E_{36}=\mu d,$
$E_{41}=0,\ E_{42}=0,\ E_{43}=0,\ E_{44}=b,\ E_{45}=0,\ E_{46}=0,$

$E_{51}=\mu,\ E_{52}=0,\ E_{53}=-\mu,\ E_{54}=0,\ E_{55}=\mu,\ E_{56}=-\mu d,$
$E_{61}=-\mu d,\ E_{62}=0,\ E_{63}=\mu d,\ E_{64}=0,\ E_{65}=-\mu d \ and\ E_{66}=c+\mu d^2$

The E_{ij}s' above are the elements of the control operator $\tilde{E}$ which can now be written in form of matrix as

$$\tilde{E}=\begin{pmatrix} \mu & 0 & -\mu & 0 & \mu & -\mu d \\ 0 & a & 0 & 0 & 0 & 0 \\ -\mu & 0 & \mu & 0 & -\mu & \mu d \\ 0 & 0 & 0 & b & 0 & 0 \\ \mu & 0 & -\mu & 0 & \mu & -\mu d \\ -\mu d & 0 & \mu d & 0 & -\mu d & c+\mu d^2 \end{pmatrix} \tag{11}$$

Therefore, (11) is the required control operator $\tilde{E}$. On factorizing, it yields

$$\tilde{E}=\mu\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & -d \\ 0 & \mu^{-1}a & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & d \\ 0 & 0 & 0 & \mu^{-1}b & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & -d \\ -d & 0 & d & 0 & -d & \mu^{-1}c+d^2 \end{pmatrix} \tag{12}$$

Let us take the limit of $\tilde{E}$, that is

$$\lim_{\mu\to\infty}\tilde{E}=\lim_{\mu\to\infty}\left\{\mu\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & -d \\ 0 & \mu^{-1}a & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & d \\ 0 & 0 & 0 & \mu^{-1}b & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & -d \\ -d & 0 & d & 0 & -d & \mu^{-1}c+d^2 \end{pmatrix}\right\}$$

$$=\lim_{\mu\to\infty}\mu\times\lim_{\mu\to\infty}\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & -d \\ 0 & \mu^{-1}a & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & d \\ 0 & 0 & 0 & \mu^{-1}b & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & -d \\ -d & 0 & d & 0 & -d & \mu^{-1}c+d^2 \end{pmatrix} \tag{13}$$

$$=\lim_{\mu\to\infty}\mu\times\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & -d \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & d \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & -d \\ -d & 0 & d & 0 & -d & d^2 \end{pmatrix} \tag{14}$$

Hence,

$$\lim_{\mu\to\infty}\left|\tilde{E}\right|=\lim_{\mu\to\infty}\mu\times\left|\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & -d \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & d \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & -d \\ -d & 0 & d & 0 & -d & d^2 \end{pmatrix}\right| \tag{15}$$
$$=\infty\times 0=0$$

Which implies,

$$\lim_{\mu\to\infty}J_\mu(v,u,w)\ =\ 0 \tag{16}$$

Hence our cost functional is bounded, Hermittian and self-adjoint, therefore solution exists and it will be superlinealy convergent if an appropriate penalty function method is applied.

3.0 Conclusion

This paper is an alternative form of showing that our control problem[3] has a unique solution. This result is simpler than our former result [6].

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