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On The Solution of Some Ordinary Differential Equations Using Adomian Decomposition Method

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Abstract

In this paper, the solution of some ordinary differential equations using Adomian Decomposition Method (ADM) is considered. The procedures and steps involves in using the Adomian Decomposition Method were thoroughly reviewed using a general typical non-linear ordinary differential operator. The deficiencies of using the alternative techniques which includes Descretization, Perturbation, and Linearization methods instead of the present one were also investigated while it was shown that this method devoid of problems and errors encountered using those methods. From the numerical examples presented, the convergence rate of the method is very high which confirms the earlier claims in the literatures.

Keywords: differential equations, Adomian decomposition, non-linear, linearization, convergence.

1.0 Introduction

It is of concern that solutions to most differential equations that arise in integral and differential models representing some phenomena in Engineering, Management, Economics and Science cannot easily be obtained using analytical means, therefore, approximate solutions are needed which are generated by numerical techniques.

The Adomian decomposition method is a relatively new approach which provides an analytic approximation to linear and non-linear problems. It is a quantitative method rather than qualitative. It requires neither linearization nor perturbation, so you don't have to worry about interpolating variables. It is free from minding off errors as it does not involve discretization. This method is useful for obtaining both a closed form and the explicit solution and the numerical approximations of linear and non-linear differential equations.

George Adomian [1], is the Armenian – American mathematician who developed the Adomian Decomposition Method (ADM) for solving both ordinary and partial non-linear differential equations. The method is explained, among other places, in his book *"Solving Frontier Problems in Physics: the Decomposition Method"*. He was a faculty member at the University of Georgia (USA) from 1966 through 1989.

This method is based on the search for a solution in the form of a series and it consists in decomposing the non-linear operator into a series. George Adomian developed the ADM in the 1980s and showed how it can be applied to solving non-linear differential equations.

Application of the method to fractional differential equations and other various fields of applied sciences are also found. An error analysis and convergence criterion of the ADM has been investigated by several authors. Cherruault [2], investigated the convergence of the method when applied to a special class of boundary valued problems of periodic temperature fields in heat conductance.

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In the recent years, Adomian decomposition method has been used to solve ordinary differential equations, differential – algebraic equation, non-linear fractional differential equation and delay differential equation. In most of these works, the convergence of the decomposition series had been investigated by several researchers [3 - 5]. Ibijola et al [6] applied Adomian Decomposition Method for numerical solution of second-order ordinary differential equations which yields some highly commendable results.

In this paper, the work in [3 - 6] is extended to ordinary differential equations of higher orders, most especially non-linear problems. Numerical examples are presented while the Euler method is also used to test the efficiency of the method.

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2.0 Developmental Procedures Of Adomian Decomposition Method

Each step is better explained in the following way

Step 1: Split the equation into linear and non-linear parts.
Step 2: Invert the highest order derivative operator contained in the linear operator on both sides.
Step 3: Identify the initial conditions and the terms involving the independent variables alone as initial approximation.
Step 4: Decompose the unknown function into a series whose components can be easily computed.
Step 5: Decompose the non-linear functions in terms of polynomials (Adomian polynomials).
Step 6; Find the successive terms of the series solution by current relations using the polynomials obtained

2.1 A Typical Example

In this section, the techniques involves in using the Adomian Decomposition Method to solve problems that can be modeled in form of the differential equations is demonstrated.

Consider the operator

$$Fu = G \qquad (1)$$

where F represents a general non-linear ordinary differential operator and G is a given function. The linear part of F is decomposed into

$$L + R \qquad (2)$$

where L is easily invertible and R is the remainder of F Thus the equation may be written as

$$Lu + Ru + Nu = G \qquad (3)$$

Where, N is a non-linear operator, L is the highest order derivative which is assumed to be invertible R is a linear differential operator of the order less than L and G is the source term

The method is based on applying the operator L^{-1} formally to the expression

$$Lu = G - Ru - Nu \qquad (4)$$

So, by using the given conditions, we obtain

$$u = h + L^{-1}G - L^{-1}Ru - L^{-1}Nu \qquad (5)$$

Where h is the solution of the homogeneous equation

$$Lu = 0, \qquad (6)$$

with the initial/boundary conditions. Now according to the decomposition procedure of Adomian, we construct the unknown function u(x) by a sum of components defined by the following decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (7)$$

The problem now, is the decomposition of the nonlinear term Nu. To do this, Adomian developed a very elegant technique as follows:

Define the decomposition parameter λ as

$$u = \sum_{n=0}^{m} \lambda^n u_n \qquad (8)$$

then $N(u)$ will be a function of ë, $u_0, u_1, ...$ next expanding N(u) in Maclaurin series with respect to ë, we obtain

$N(u) = \sum_{n-0}^{\infty} \lambda^n A_n$, where

$$A = \frac{1}{n!}\left[\frac{d}{d\lambda^n}\ N\left(\sum_{k=0}^{n} \lambda^k u_k\right)\right]_{\lambda=0}$$

$$A_0 = f(u_0)$$

$$A_1 = u_1 f'(u_0) + \frac{1}{2!} {u_1}^2 f^{(2)}(u_0) + \frac{1}{3!} {u_1}^3 f^{(3)}(u_0) + \cdots$$

$$A_2 = u_2 f'(u_0) + \frac{1}{2!}(u_2^2 + 2u_1u_2) f''(u_0) + \frac{1}{3!}(3u_1^2 u_2 + 3u_1 u_2^2 + u_2^3) f'''(u_0) + \cdots$$

$$A_3 = u_3 f'(u_0) + \frac{1}{2!}(u_3^2 + 2u_1u_3 + 2u_2u_3) f''(u_0) + \frac{1}{3!}(u_3^3 + 3(u_1 + u_2) + 3u_3(u_1 + u_2)^2) f'''(u_0) + \cdots \quad (9)$$

Based on the Adomian decomposition method, the solution u is constructed as

$$u = \lim_{n \to \infty} \phi_n \quad (10)$$

where the (n+1)-term approximation of the solution is defined in the following form:

$$\phi_{n+1} = \sum_{k=0}^{n} u_k(x) \quad (11)$$

The method reduces significantly the massive computation which may arise if discretization methods are used for the solution of non non-linear problems.

3.0 Alternative Techniques

The three main alternative methods that can be used to solve non-linear differential equations are discussed in this section. These includes discretization, perturbation, and linearization methods

Some of the existing methods to solving ordinary differential equations are based on discretization and they only allow the solutions to a given ordinary differential equation at a given interval. The above deficiency leads to a situation where some fundamental phenomena are easily avoided. However, discretization concerns the process of transferring continuous models and equations into discrete counterparts. We see this in the Euler's method, Runge-kutta, Adam-Bosworth's method of solving differential equations amongst others. These methods can be very tedious because of the large amounts of values that need to be calculated.

In perturbation theory, a set of mathematical methods for obtaining approximate solutions to complex equations for which no exact solution is possible or known, generally involving an iterative algorithm in which each new term contributing to the solution has less significance than the last. In a physical situation, an unknown quantity is required to satisfy a given differential equation and certain auxiliary conditions that define the values of the unknown quantity at specified times or positions. If the equation or auxiliary conditions are varied slightly, the solution to the problem will also vary slightly. Perturbation is a method for solving a problem by comparing it with a similar one for which the solution is known. Usually the solution found in this way is only approximate.

In Mathematics, solving non-linear differential equations is a very important aspect. Most of the methods that are available and easy to use are methods for solving linear differential equations. Nonlinear differential equations are usually arising from mathematical modeling of many frontier physical systems. In most cases, analytic solutions of these differential equations are very difficult to achieve. Common analytic procedures are used to linearize the system or assume the nonlinearities are relatively insignificant. Such procedures change the actual problem to make it tractable by the conventional methods. Generally, the numerical methods such as Runge–Kutta method are based on discretization techniques, and they only permit us to calculate the approximate solutions for some values of time and space variables, which causes us to overlook some important phenomena such as chaos and bifurcation, in addition to the intensive computer time required to solve the problem. The above drawbacks of linearization and numerical methods arise the need to search for an alternative techniques to solve the nonlinear differential equations. Throughout, we shall consider equation of the form;

$$y''' = f(x, y), y(0) = y_0, y'(0) = y_1, y''(0) = y_2. \quad (12)$$

4.0 Illustrative Examples

4.1 Example 1

Consider the differential equation

$$y''' + 2xy'' + 4y' - x^2 y = 1 \quad (13)$$

With the initial conditions

$$y''(0) = 3, y'(0) = 2, y(0) = 1 \quad (14)$$

Rewriting (13), we have

$$y''' = 1 + x^2 y - 4y' + 2xy'' \quad (15)$$

Writing in operator form

$$Ly = 1 + x^2 y - 4y' + 2xy'' \quad (16)$$

Where $L = \dfrac{d^3}{dx^3}$

$$L^{-1} = \int_0^x\int_0^x\int_0^x (.)dxdxdx$$

Applying the inverse operator to equation (16),

$$L^{-1}(Ly) = h + L^{-1}(1) + L^{-1}(x^2 y) - L^{-1}4\frac{d}{dx}y + L^{-1}(2x\frac{d^2}{dx^2}y) \qquad (17)$$

Where h is the solution of the homogeneous equation $Ly = 0$ with the initial conditions, subsequently, (17) gives

$$y(x) = 1 + 2x + 3x^2 + L^{-1}(1) + L^{-1}x^2(y) - L^{-1}4\frac{d}{dx}(y) + L^{-1}2x\frac{d^2}{dx^2}(y) \qquad (18)$$

Using the Adomian decomposition procedure,

$$y = \sum_{n=0}^{\infty} y_n \qquad (19)$$

Substituting (19) into (18), yields

$$\sum_{n=0}^{\infty} y_n(x) = 1 + 2x + 3x^2 + L^{-1}(1) + L^{-1}x^2\left(\sum_{n=0}^{\infty} y_n\right) - L^{-1}4\frac{d}{dx}\left(\sum_{n=0}^{\infty} y_n\right) + L^{-1}2x\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} y_n\right) \qquad (20)$$

We must state here that in practice, all terms of the series in (19) cannot be determined and the solution will be approximated by series of the form [3] and [6]

$$\Phi_n(x) = \sum_{n=0}^{n-1} y_n(x) \qquad , n \geq 0 \qquad (21)$$

The method introduces the recursive relation

$$y_0(x) = 1 + 2x + 3x^2 + L^{-1}(1) = 1 + 2x + 3x^2 + \frac{x^3}{6}$$

$$y_{n+1}(x) = L^{-1}x^2(y_n) - L^{-1}4\frac{d}{dx}(y_n) + L^{-1}2x\frac{d^2}{dx^2}(y_n) \qquad (22)$$

$$n \geq 0$$

We can the proceed to compute the first few terms of the series.

$$y_1(x) = L^{-1}x^2 y_0 - L^{-1}4\frac{d}{dx}y_0 + L^{-1}2x\frac{d^2}{dx^2}y_0$$

$$= L^{-1}x^2\left(1 + 2x + 3x^2 + \frac{x^3}{6}\right) - L^{-1}4\frac{d}{dx}\left(1 + 2x + 3x^2 + \frac{x^3}{6}\right) + L^{-1}2x\frac{d^2}{dx^2}\left(1 + 2x + 3x^2 + \frac{x^3}{6}\right)$$

$$= L^{-1}\left(x^2 + 2x^3 + 3x^4 + \frac{x^5}{6}\right) - L^{-1}4\left(2 + 6x + \frac{1}{2}x^2\right) + L^{-1}2x(6 + x)$$

$$= L^{-1}\left(x^2 + 2x^3 + 3x^4 + \frac{x^5}{6}\right) - L^{-1}(8 + 24x + 2x^2) + L^{-1}(12x + 2x^2)$$

$$= L^{-1}\left(x^2 + 2x^3 + 3x^4 + \frac{x^5}{6} - 8 - 24x - 2x^2 + 12x + 2x^2\right)$$

$$=\int_0^x\int_0^x\int_0^x\left(\frac{x^5}{6}+3x^4+2x^3+x^2-12x-8\right)dxdxdx$$

$$=\int_0^x\int_0^x\left(\frac{x^6}{36}+\frac{3x^5}{5}+\frac{2x^4}{4}+\frac{x^3}{3}-\frac{12x^2}{2}-8x\right)dxdx$$

$$=\int_0^x\left(\frac{x^7}{252}+\frac{3x^6}{30}+\frac{2x^5}{20}+\frac{x^4}{12}-\frac{12x^3}{6}-\frac{8x^2}{2}\right)dx$$

$$=\left(\frac{x^8}{2016}+\frac{3x^7}{210}+\frac{2x^6}{120}+\frac{x^5}{60}-\frac{12x^4}{24}-\frac{8x^3}{6}\right)$$

$$y_2(x)=L^{-1}x^2y_1-L^{-1}4\frac{d}{dx}y_1+L^{-1}2x\frac{d^2}{dx^2}y_1$$

$$=L^{-1}x^2\left(\frac{x^8}{2016}+\frac{3x^7}{210}+\frac{2x^6}{120}+\frac{x^5}{60}-\frac{12x^4}{24}-\frac{8x^3}{6}\right)-L^{-1}4\frac{d}{dx}\left(\frac{x^8}{2016}+\frac{3x^7}{210}+\frac{2x^6}{120}+\frac{x^5}{60}-\frac{12x^4}{24}-\frac{8x^3}{6}\right)$$

$$+L^{-1}2x\frac{d^2}{dx^2}\left(\frac{x^8}{2016}+\frac{3x^7}{210}+\frac{2x^6}{120}+\frac{x^5}{60}-\frac{12x^4}{24}-\frac{8x^3}{6}\right)$$

$$=L^{-1}\left(\frac{x^{10}}{2016}+\frac{3x^9}{210}+\frac{2x^8}{120}+\frac{x^7}{60}-\frac{12x^6}{24}-\frac{8x^5}{6}\right)-L^{-1}\left(\frac{32x^7}{2016}+\frac{84x^6}{210}+\frac{48x^5}{120}+\frac{20x^4}{60}-\frac{192x^3}{24}-\frac{96x^2}{6}\right)$$

$$+L^{-1}\left(\frac{112x^7}{2016}+\frac{252x^6}{210}+\frac{120x^5}{120}+\frac{40x^4}{80}-\frac{288x^3}{24}-\frac{96x^2}{6}\right)$$

$$=L^{-1}\left(\frac{x^{10}}{2016}+\frac{3x^9}{210}+\frac{2x^8}{120}+\frac{x^7}{60}+\frac{80x^7}{2016}-\frac{12x^6}{24}+\frac{168x^6}{210}-\frac{8x^5}{6}+\frac{72x^5}{120}+\frac{20x^4}{60}-\frac{96x^3}{24}\right)$$

$$=\int_0^x\int_0^x\int_0^x\left(\frac{x^{10}}{2016}+\frac{3x^9}{210}+\frac{2x^8}{120}+\frac{x^7}{60}+\frac{80x^7}{2016}-\frac{12x^6}{24}+\frac{168x^6}{210}-\frac{8x^5}{6}+\frac{72x^5}{120}+\frac{20x^4}{60}-\frac{96x^3}{24}\right)dxdxdx$$

$$=\int_0^x\int_0^x\left(\frac{x^{11}}{22176}+\frac{3x^{10}}{2100}+\frac{2x^9}{1080}+\frac{x^8}{480}+\frac{80x^8}{16128}-\frac{12x^7}{168}+\frac{168x^7}{1470}-\frac{8x^6}{36}+\frac{72x^6}{720}+\frac{20x^5}{300}-x^4\right)dxdx$$

$$=\int_0^x\left(\frac{x^{12}}{266112}+\frac{3x^{11}}{23100}+\frac{2x^{10}}{10800}+\frac{x^9}{4320}+\frac{8x^9}{145152}-\frac{12x^8}{1344}+\frac{16x^8}{11760}-\frac{8x^7}{252}+\frac{72x^7}{5040}+\frac{20x^6}{1800}-\frac{x^5}{5}\right)$$

$$=\frac{x^{13}}{3459456}+\frac{3x^{12}}{277200}+\frac{2x^{11}}{118800}+\frac{x^{10}}{43200}+\frac{8x^{10}}{1451520}-\frac{12x^9}{12096}+\frac{16x^9}{105840}-\frac{8x^8}{2016}+\frac{72x^8}{40320}+\frac{20x^7}{12600}-\frac{x^6}{30}$$

Therefore,

$$y(x)=1+2x+3x^2+\frac{x^3}{6}-\frac{8x^3}{6}-\frac{12x^4}{24}+\frac{x^5}{60}+\frac{2x^6}{120}-\frac{x^6}{30}+\frac{3x^7}{210}+\frac{20x^7}{12600}+\frac{x^8}{2016}+\frac{72x^8}{40320}-\frac{8x^8}{2016}+$$

$$\frac{16x^9}{10580}-\frac{12x^9}{12096}+\frac{8x^{10}}{1451520}+\frac{x^{10}}{43200}+\frac{2x^{11}}{118800}+\frac{3x^{12}}{277200}+\frac{x^{13}}{3459456}+\ldots$$

Table 1 compares the result obtained using ADM with the Euler's solution. It is obvious that the result is in agreement with the Euler's solution. Higher accuracy can be obtained by evaluating more components of the series in equation (20).

Table 1: Adomian and Euler method

X	ADOMIAN	EULER	ERROR
0.0	1.000000000	1.000000000	0.000000000
0.1	1.200000000	1.200000000	0.000000000
0.2	1.4300000333	1.430000000	0.000000333
0.3	1.6834481711	1.683000000	0.0004481711
0.4	2.192660635	2.1926532500	0.000007385
0.5	2.573294503	2.573928280	0.000633777

4.2 Example 2

Consider the differential equation

$$y'''+y''+y'+y=0$$

$$y''(0)=0, y'(0)=0, y(0)=1$$

$$rewritting,$$

$$y'''=-y-y'-y''$$

$$Ly=-y-y'-y'' \tag{23}$$

The exact solution of equation (23) is

$$L=\frac{d^3}{dx^3}$$

$$L^{-1}=\int_0^x\int_0^x\int_0^x(.)dxdxdx$$

Applying the inverse operator to the equation,

$$L^{-1}(Ly)=y_0+y_1x+y_2x^2-L^{-1}(y)-L^{-1}(y)'-L^{-1}(y'') \tag{24}$$

Note: $y_0+y_1x+y_2x^2$ is obtained by solving the homogeneous equation $Ly=0$

Now, from the initial values, we obtain $y_0=1, y_1=0, y_2=0$ (25)

So we have from (24)

$$y=1-L^{-1}(y)-L^{-1}(y')-L^{-1}(y'') \tag{26}$$

Substitute $y''=\frac{d^2}{dx^2}, y'=\frac{d}{dx}$ into (26)

$$y=1-L^{-1}(y)-L^{-1}\frac{d}{dx}(y)-L^{-1}\frac{d^2}{dx^2}(y) \tag{27}$$

Using the decomposition series, we have

$$y(x)=\sum_{n=0}^{\infty}y_n(x)$$

So we have,

$$\sum_{n=0}^{\infty}y_n(x)=1-L^{-1}\left(\sum_{n=0}^{\infty}y_n\right)-L^{-1}\frac{d}{dx}\left(\sum_{n=0}^{\infty}y_n\right)-L^{-1}\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty}y_n\right)$$

So we have the recursive relation,

$$y_0 = 1$$

$$y_{n+1} = -L^{-1}(y_n) - L^{-1}\frac{d}{dx}(y_n) - L^{-1}\frac{d^2}{dx^2}(y_n)$$

$$y_1 = -L^{-1}(y_0) - L^{-1}\frac{d}{dx}(y_0) - L^{-1}\frac{d^2}{dx^2}(y_0)$$

$$= -L^{-1}(1) - L^{-1}\frac{d}{dx}(1) - L^{-1}\frac{d^2}{dx^2}(1)$$

$$= -L^{-1}(1)$$

$$= \frac{x^4}{4!}$$

$$y_2 = -L^{-1}(y_1) - L^{-1}\frac{d}{dx}(y_1) - L^{-1}\frac{d^2}{dx^2}(y_1)$$

$$= -L^{-1}\left(\frac{x^4}{4!}\right) - L^{-1}\frac{d}{dx}\left(\frac{x^4}{4!}\right) - L^{-1}\frac{d^2}{dx^2}\left(\frac{x^4}{4!}\right)$$

$$= L^{-1}\left(\frac{x^4}{4!}\right) - L^{-1}\left(\frac{x^3}{3!}\right) - L^{-1}\left(\frac{x^2}{2}\right)$$

$$= L^{-1}\left(-\frac{x^4}{4!} - \frac{x^3}{3!} - \frac{x^2}{2}\right)$$

$$= \int\int\int\left(-\frac{x^4}{4!} - \frac{x^3}{3!} - \frac{x^2}{2}\right)$$

$$= -\frac{x^7}{7!} - \frac{x^6}{6!} - \frac{x^5}{5!}$$

$$y(x) = 1 - \frac{x^4}{4!} - \frac{x^7}{7!} - \frac{x^6}{6!} - \frac{x^5}{5!} + \ldots$$

Table 2 compares the result obtained using ADM with that of the Euler's method, which is one of the approximated numerical method. It is clear that the Adomian decomposition method works very well. Errors are small and may be made smaller by using more terms of the ADM truncated series.

Table 2: Adomian and Euler methods

X	Adomian	Euler	Error
0.0	1.000000000	1.000000000	0.000000000
0.1	0.999993666	1.000000000	0.000006334
0.2	0.999930574	1.000000000	0.000069426
0.3	0.999641194	0.999000000	0.000641194
0.4	0.998841986	0.996100000	0.002741986
0.5	0.997112165	0.990500000	0.006612165

5.0 Convergence of the Method

The decomposition series (7) solution is generally converges very rapidly in real physical problems [1]. The rapidity of this convergence means that few terms are required. Convergence of this method has been rigorously established [2- 5].

6.0 References

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