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Convergence of Numerical Solution for Heat Equation

Augustine O. Odio

Department of Mathematics,
University of Nigeria, Nsukka

Abstract

The work is on convergence of numerical solution for heat equation. In this problem, we consider a suitable difference scheme whose dependent variable u is the control which depends on multiplicity of space variables $x_1, x_2, x_3 \ldots, x_n$ and time variable t. Here, u is defined on suitable subspaces of the space of definition. This type of function u, is said to be admissible and also, satisfies the Taylor series expansions.

Also, the difference scheme in question satisfy the numerical properties such as consistency, stability and convergent. Numerical solution obtained were found to be constant, stable and convergent.

Keywords: Convergence, numerical solution, numerical scheme, consistency, stability admissible function.

1.0 Introduction

Consider the heat equation

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2} \tag{1.1}$$

where t and x are the time and distance coordinates respectively, in the region $\Re = [0 \le x \le 1] \times [t \ge 0]$, with appropriate initial and boundary conditions. The region $\Re$ is replaced by a set of points $\Re_i$ which are the vertices of grid points (m,n)where x = a + mb,
t = nk with Mh = b – a, M being integer. The quantities k and h are mesh sizes in the time and space direction respectively.

In equation (1.1) u is taken as the control, a quantity that adds heat per unit time of the system, c is the wave speed (see [1] and [2])

We write the finite difference scheme for equation (1.1) as

$$\frac{u_{i,n+1} - u_{i,n}}{\Delta t} = \frac{u_{i-1,n} - 2u_{i,n} + u_{i+1,n}}{(\Delta x)^2} \tag{1.2}$$

where n is the subscript of space at time level [3]. This is the explicit form of the finite difference equation.

In equation (1.2) the control term u depends on multiplicity of space variables x_1, x_2, $x_3 \ldots$, x_n and time t in the space of definition. This type of function u is set to be admissible [4] and hence, it admit the Taylor series expansion [5] which satisfies equation (1.1) as expected.

We then write equation (1.2) as

$$u_{i,n+1} - u_{in} = \frac{\Delta t}{(\Delta x)^2}\left[u_{i+1,n} - 2u_{i,n} + u_{i-1,n}\right] \tag{1.3}$$

Let $\dfrac{\Delta t}{(\Delta x)^2} = \lambda$ (1.4)

$$u_{i,n+1} - u_{i,n} = \lambda u_{i+1,n} - 2\lambda u_{i,n} + \lambda u_{i-1,n} \tag{1.5}$$

$$u_{i,n+1} = \lambda u_{i+1,n} + (1-2\lambda)\, u_{i,n} + \lambda u_{i-1,n} \tag{1.6}$$

A practical result for convergence of the numerical scheme for the solution of the parabolic equation is given in the equivalent theorem of Lax.

Corresponding author: E-mail: augustine.odio@yahoo.com, Tel. +234 8062976038

Equivalent Theorem of Lax [6]: For a well posed linear, initial value problem with a consistent discretization stability is the necessary and sufficient condition for convergence of the numerical scheme.

2.0 Main Results

2.1 Consistency

We consider the numerical scheme

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \tag{2.1}$$

for the solution of the heat equation of (1.1)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{\Delta t}\left[u(x,t) + \Delta t\, u_t(x,t) + \frac{(\Delta t)^2}{2} + u_{tt}(x,t)\ldots - u(x,t)\right] \tag{2.2}$$

$$= u_t + \frac{\Delta t}{2} u_{tt} \ldots. + \tag{2.3}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - u_t = \frac{\Delta t}{2} u_{tt} + \ldots.. = o,\ (\Delta t) \longrightarrow 0 \tag{2.4}$$

$$u_{i+1} = u(x,t) + \Delta x u_x^n(x,t) + \frac{(\Delta x)^2}{2} u_{xx}^n(x,t) + \frac{(\Delta x)^3}{3!} u_{xxx}(x,t) + \ldots \tag{2.5}$$

$$u_{i-1}^n = u(x,t) - \Delta x u_x^n(x,t) + \frac{(\Delta x)}{2!} u_{xx}(x,t) - \frac{(\Delta x)^3}{3!} u_{xxx}^n(x,t) + \ldots \tag{2.6}$$

$= 0\,(\Delta x)^2$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x^2} - (u_t - u_{xx}) = O(\Delta t) + O(\Delta x)^2 \tag{2.7}$$

Since the total energy will approach zero as $\Delta t \to 0$ and $\Delta x \to 0$, we say that the scheme is consistent for the parabolic equation it is supposed to solve.

2.2 Stability

We consider the numerical scheme

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \tag{2.8}$$

Let $\lambda = \dfrac{\Delta t}{(\Delta x)^2}$

$$u_i^{n+1} - u_i^n = \lambda(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

$$u_i^{n+1} = \lambda u_{i+1}^n + (1-2\lambda)u_i^n + \lambda u_{i-1}^n \tag{2.9}$$

Put $u_i^n = \phi(t)e^{i\alpha x}$

$$u_i^{n+1} = \phi(t+\Delta t)e^{i\alpha x}$$

$$\phi(t+\Delta t)e^{i\alpha x} = \lambda\phi(t)e^{i\alpha(x+\Delta x)} + (1-2\lambda)\phi(t)\, e^{i\alpha x} + \lambda\phi(t)\, e^{i\alpha(x-\Delta x)} \tag{2.10}$$

$$= \phi(t)[\lambda e^{i\alpha\Delta x}\, e^{i\alpha x} + (1-2\lambda)e^{i\alpha x} + \lambda e^{i\alpha x}\, e^{-i\alpha x}] \tag{2.11}$$

$$= \phi(t)\,[(1-2\lambda) + \lambda e^{i\alpha\Delta x} + e^{-i\alpha\Delta x})] \tag{2.12}$$

$$= \phi(t)\,[(1-2\lambda) + \lambda\,(\cos\alpha\Delta x + i\sin\alpha\,\Delta x + \cos\alpha\Delta x - i\sin\alpha\,\Delta x] \tag{2.13}$$

$$= \phi(t)\,[(1-2\lambda) + 2\lambda\,(\cos^2\alpha\,\frac{\Delta x}{2} - \sin^2\alpha\,\frac{\Delta x}{2})]$$

$$= \phi(t)\,[(1-2\lambda) + 2\lambda(1 - 2\sin^2\alpha\frac{\Delta x}{2})]$$

$$= \phi(t)[1 - 2\lambda + 2\lambda - 4\lambda\sin^2\alpha\,\frac{\Delta x}{2}]$$

$= \phi(t)\ [1-4\lambda\sin^2 \alpha \frac{\Delta x}{2}]$ (2.14)

$\frac{\phi(t+\Delta t)}{\phi(t)} = 1-4\lambda s^2$ (2.15)

where $s = \sin \alpha \frac{\Delta x}{2}$

For stability we must have

$\left|\frac{\phi(t+\Delta t)}{\phi(t)}\right| \leq 1,$

that is, $\left|\frac{\phi(t)+\Delta t)}{\phi(t)}\right| = |1-4\lambda s^2| \leq 1$ (2.16)

which implies that

that is,

$|1 < 1-4\lambda s^2 < 1$

that is, $-2 < -4\lambda s^2$ and $-4\lambda s^2 < 0$

that is, $4\lambda s^2 < 2$, $4\lambda\ s^2 > 0$

$\lambda s^2 \leq ½$

Since $|s^2|^2 = 1$, we have that

$\lambda\ \leq ½$ and $\lambda > 0$

$= 0 \leq \lambda \leq ½$ (2.17)

Since the numerical scheme is consistent and stable it implies that, the scheme converges. Following the result in equation (2.17), we give a concrete application for convergence.

Consider the heat flow problem $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ with boundary conditions

$u(0,t) = 0 = u(8,t)$ and $u(x,0) = 4x- \frac{1}{2^2}x^2$ at the points x = 1, t = 0,

i = 1,2,3,…,7 and $t = \frac{1}{8}$ j: j = 0,1,2,…,5. Find the value of u(x,t).

Solution

$c^2 = 4$, $h = 1$, $k = \frac{1}{8}$

$a = \frac{c^2 k}{h^2} = \frac{4.\frac{1}{8}}{1} = \frac{1}{2}$

Then the equation is

$u_{i,\ j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j})$ (2.18)

Given $u(0,t) = u(8,t) = 0$

or $u(_{0,j}) = 0 = u(8,_j)\ \forall\ j = 1,2,3,4,5$

and $u(x,0) = 4x - \frac{1}{2}x^2 = u_{i,\ 0} = 4_i - \frac{1}{2}i^2$

$u_{0,0} = 0$, $u_{1,0} = 4(1) - \frac{1}{2}(1)^2 = 3.5$, $u_{2,0} = 4(2) - \frac{1}{2}(2)^2 = 6$

$u_{3,0} = 7.5$, $u_{4,0} = 8$, $u_{5,\ 0} = 7.5$, $u_{6,0} = 6$, $u_{7,\ 0} = 3.5$

Putting j = 0 in (2.18) we get

$u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$

$u_{1,1} = \frac{1}{2}(u_{0,0}+u_{2,0}) = \frac{1}{2}(0+6) = 3$, $u_{2,1} = \frac{1}{2}(u_{1,0}+u_{3,0})\ \frac{1}{2}(3.5+7.5) = 5.5$

$u_{3,1} = \frac{1}{2}(u_{2,0}+u_{4,0}) = \frac{1}{2}(6+8)\ 7$, $u_{4,1} = \frac{1}{2}(u_{3,0}) + u_{5,0}) = \frac{1}{2}(7.5 + 7.5) = 7.5$

$u_{5,1} = \frac{1}{2}(u_{4,0}) + u_{6,0}) = \frac{1}{2}(8+6) = 7$, $u_{6,1} = \frac{1}{2}(u_{5,0}+u_{7,0}) = \frac{1}{2}(7.5+3.5) = 5.5$

$u_{7,1} = \frac{1}{2}(u_{6,0}+u_{8,0}) = \frac{1}{2}(6+0) = 3$

Putting j = 1 in (2.18) we have

$u_{i,2} = ½ (u_{i-1,1}+u_{i+1,1})$

$u_{1,2} = \frac{1}{2}(u_{0,1}+u_{2,0}) = \frac{1}{2}(0+5.5) = 2.75$, $u_{2,2} = \frac{1}{2}(u_{1,1}+u_{3,1}) = \frac{1}{2}(3+7) = 5$

$u_{3,2} = 6.5$, $u_{4,2} = 7$, $u_{5,} = 6.5$, $u_{6,2} = 5$, $u_{7,2} = 2.75$

Putting j = 2 in (2.18) we have

$u_{i,3} = ½ (u_{i-1,2}+u_{i+1,2})$

$u_{1,3} = \frac{1}{2}(u_{0,2}+u_{2,2}) = \frac{1}{2}(0.5) = 2.5$, $u_{2,3} = \frac{1}{2}(u_{1,2+}u_{3,2} = \frac{1}{2}(2.75+6.5) = 4.625$

$u_{3,3} = 6$, $u_{4,3} = 6.5$, $u_{5,3} = 6$, $u_{6,3} = 4.625$, $u_{7,3} = 2.5$

putting j = 3 in (2.18) we have

$u_{i,4} = ½ (u_{i-1,3}+u_{i+1,3})$

$u_{1,4} = \frac{1}{2}(u_{0,3}+u_{2,3}) = \frac{1}{2}(0+4.625) = 2.3125$, $u_{2,4} = \frac{1}{2}(u_{1,3}+ u_{3,3+}) = \frac{1}{2}(2.5+6) = 4.25$

$u_{3,4} = 0.3625$, $u_{4,4} = 6$, $u_{5,4} = 5.5625$, $u_{6,4} = 4.25$, $u_{7,4} = 2.3125$

Putting j = 4 in (2.18) we have

$u_{i,5} = \frac{1}{2}(u_{i-1,4} + u_{i+1,4})$

$u_{1,5} = \frac{1}{2}(u_{0,4}+u_{2,4}) = \frac{1}{2}(0 + 4.25) = 2.125$, $u_{2,5} = \frac{1}{2}(u_{1,4}+u_{3,4}) = \frac{1}{2}(2.125 + 5.5625)\ 5.9375$

Table (2.1) Convergence of numerical solution for heat equation

j \ i	0	1	2	3	4	5	6	7	8
0	0	3.5	0	7.5	8	7.5	6	3.5	0
1	0	3	5.5	7	7.5	7	5.5	3	0
2	0	2.75	5	6.5	7	6.5	5	2.75	0
3	0	2.5	4.626	6	6.5	6	4.625	2.5	0
4	0	2.3125	4.25	5.5625	6	5.5625	4.25	2.3125	0
5	0	2.125	3.9375	5.125	5.5625	5.125	3.9375	2.125	0

3.0 Conclusion:

The numerical scheme is consistent, stable and convergent. Hence the numerical results obtained converge.

References

[1] Jain, M.K, Numerical solution of partial Differential Equations, Wiley Eastern Limited, New Delly Bargalore, Bombay, pp 213, 1978.

[2] Tejumola, H.O; Periodic Boundary Value problems for some fifth, fourth and third order ordinary differential equations. J. Nigerian math, Soc, Volume 25, pp. 37- 46 2006.

[3] K.W. Morton, and David Mayers; Numerical Solution of Partial Differential Equations, Cambridge University Press. Pp.14, (2005)

[4] Augustine O. Odio, Numerical Computation of Optimal Solution of one dimensional wave equations with diffusion effect, International Journal of Numerical Mathematics, Vol. 5, No. 1, pp 66-83; 2010.

[5] R. Barrio, F. Blesa and M. Lara, High-Precision numerical solution of ODE with high-order Taylor methods in parallel, monografias de la Real Academia de Clencias de Zaragoza, 22: 67-74, 2003.

[6] Jain, M.K, Numerical Solution of Differential Equations, Wiley Eastern Limited, New Delhi, Bangalore, Bombay, pp 212. 1978