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Stability of Numerical Solution for Wave Equation

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Abstract

The stability of numerical solution for wave equation is studied. We consider also a multi-level difference scheme in the form

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

and also, show that the scheme is stable in accordance to Von-Neumann condition for stability. In this paper, it is seen that, the numerical results become closer to each other as the i and j terms become large. Hence the computational result in Table 3.1 is stable.

Keywords: Stability, multi-level difference scheme, Von Neumann, difference equation, central difference scheme and symmetric matrix.

1.0 Introduction

We consider the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{c^2 \partial^2 u(x,t)}{\partial x^2} \tag{1.1}$$

in the domain R= $(0 \le x \le 1) \times [t \ge 0]$
satisfying the following initial conditions

$u(x,0) = f_1(x)$
$u(x,0) = f_2(x)$ for $0 \le x \le 1$
and boundary conditions
$u(0,t) = g_1(t)$
$u(1,t)= g_2(t)$ for all $t \ge 0$ (1.2)

where u is a function that depends on the space and time directions and c is the speed of the wave (see [1] and [2])
We consider also a two level scheme

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \frac{u_{j+1}^n - u_{j-i}^n}{2\Delta x} \tag{1.3}$$

which is usually called the central difference scheme [3]. The function u is continuous and differentiable in its domain R. hence u is said to be an admissible function [4] and also, u can be expanded using the Taylor series method [5].
A practical result for stability criteria for multi-level difference scheme for the solution of wave equation is given in a proposition due to Von Neumann.
Proposition (Von Neumann [6]): If $\lambda(\Delta t,k)$ is an eigenvalue of the amplification matrix G $(\Delta t,k)$ of a difference scheme, then the necessary and sufficient condition for stability are

i. $|\lambda| \le 0\,(\Delta t)$
ii. $G(\Delta t,k)$ is a symmetric matrix
iii. The scheme involves only one dependent variable.

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2.0 Main Result

We consider the difference scheme

$$\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t}=\beta\left[\frac{u_{j+1}^n-u_j^{n-1}-u_j^{n+1}+u_{j-1}^n}{(\Delta x)^2}\right] \tag{2.1}$$

$$u_j^{n+1}-u_j^{n-1}=\frac{2\Delta t\beta}{(\Delta x)^2}\left[u_{j+1}^n-u_j^{n+1}-u_j^{n-1}+u_{j-1}^n\right] \tag{2.2}$$

put $\alpha=\dfrac{2\Delta t\beta}{(\Delta x)^2}$ (2.3)

$$u_j^{n+1}-u_j^{n-1}=\alpha\left[u_{j+1}^n-u_j^{n+1}-u_j^{n-1}+u_{j-1}^n\right] \tag{2.4}$$

Let
$$\left.\begin{aligned} u_j^n &= \lambda^n e^{ikx}\\ u_j^{n+1} &= \lambda^{n+1}e^{ikx}\\ u_{j+1}^{n+1} &= \lambda^{n+1}e^{ik(x+\Delta x)}\end{aligned}\right\} \tag{2.5}$$

Substituting in (2.5) in (2.4) we have

$$\lambda^{n+1}e^{ikx}-\lambda^{n-1}e^{ikx}=\alpha[\lambda^n e^{ik(x+\Delta x)}-\lambda_{n+1}e^{ikx}-\lambda^{n-1}e^{ikx}+\lambda^n e^{ikx} \tag{2.6}$$

$$\lambda^2-1=\alpha[\lambda e^{ik\Delta x}-\lambda^2-1+\lambda e^{ik\Delta x}] \tag{2.7}$$

$$\lambda^2-1=-\alpha\lambda^2-\alpha+\alpha\lambda(e^{ik\Delta x}+e^{-ik\Delta x}) \tag{2.8}$$

$$=\alpha\lambda^2-\alpha+2\alpha\lambda\cos k\Delta x \tag{2.9}$$

Equation (2.9) becomes

$$(1+\alpha)\lambda^2-2\alpha\lambda c-(1-\alpha)=0 \tag{2.10}$$

where c = coskΔx

$$\lambda^2-\frac{2\alpha c\lambda}{1+\alpha}-\frac{(1-\alpha)}{(1+\alpha)}=0 \tag{2.11}$$

For stability condition

$|\lambda|\le 1,$

and also by considering the quadratic form

$x^2 - 2bx+c = 0$

the following conditions are in order

(i) $|b|\le 1$

(ii) $|c|\le 1$

Hence, $\dfrac{\alpha c}{1+\alpha}=\dfrac{\alpha}{1+\alpha}<1,\forall\alpha$

that is, we must have

$$|\lambda_{1,.}\lambda_2|=\left|\frac{1-\alpha}{1+\alpha}\right|\le 1,\forall\alpha \tag{2.12}$$

also, $|b|=\left|\dfrac{\alpha c}{1+\alpha}\right|\le\left|\dfrac{\alpha}{1+\alpha}\right|\le 1,\forall\alpha$

The scheme is always stable or uncondition stable.

3.0 Application

We consider the wave problem in the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \text{ with conditions}$$

u(0,t) = u(1,t) = 0, u(x,0) = ½ x(1-x) and u(x,0) = 0, taking h = 0.1 for $0 \le x$ 0.4.

Solution

$c^2 = 1$. The differences equation for the given equation is

$$u_{i,j+1} = 2(1 - \propto^2)u_{i,j} + \propto^2 \left(u_{i-1} + u_{i+1,j}\right) - u_{i,j-1} \tag{3.1}$$

Where $\alpha = \frac{k}{h}$. But $\alpha = \frac{0.1}{0.1} = 1$

Equation (3.1) reduce to

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \tag{3.2}$$

u(0,t) = u(1,t) = 0, $u_{0,j} = 0$ and $u_{10,j} = 0$

That is, the entries in the first column are all zero.

Hence, u(x,0) = ½ x(1-x), u(i,0) = ½ i(1-i) (3.3)

= 0.045, 0.08, 0.105,0.120,0.125,0.120,0.105

for i = 0.1, 0.2,0.3, 0.4, 0.5, 0.6, 0.7 at t = 0

these are the entries of the first row.

Since $u_x(x,0) = 0$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = 0 \text{ for } j = 0,\ t = 0, u_{i,1} = u_{i,9} \tag{3.4}$$

Putting j = 0 in equation (3.2) we get

$u_{i,1} = u_{i-1,\,0} + u_{i+1,\,0} - u_{i,-1}$ (3.5)

$= u_{i-1,\,0} + u_{i+1,0} - u_{i,1}$

$2u_{i,1} = u_{i-1,0} + u_{i+1,\,0}$, $u_{i,1} = ½\,[u_{i-1,0} + u_{i+1,0}]$ (3.6)

for i = 1, $u_{1,1} = ½\,[u_{0,0} + u_{2,0}] = ½\,[0+.080] = 0.040$

i = 2, $u_{2,1} = ½\,[u_{1,0} + u_{3,0}] = ½\,[0.045+0.105] = 0.075$

i = 3, $u_{3,1} = ½\,[u_{2,0} + u_{4,0}] = ½\,[0.08+0.120] = 0.100$

i = 4, $u_{4.1} = ½\,[u_{3,0} + u_{5,0}] = ½\,[0.105+0.125] = 0.115$

i = 5, $u_{5,1} = ½\,[u_{4,0} + u_{6,0}] = ½\,[0.120 + 0.120] = 0.120$

i = 6, $u_{6,1} = ½\,[u_{5,0} + u_{7,0}] = ½\,[0.125+0.105] = 0.115$

putting j = 1 in equation (3.2), we get

$$u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$$

For i = 1, $u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0}$

= 0+0.075 – 0. 0.045 = 0.03

i =2, $u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 0.040 + 0.100 - 0.08 = 0.060$

i =3, $u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 0.075+0.115-0.105 = 0.085$

i =4, $u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = 0.100+0.120-0.120 = 0.100$

i =5, $u_{5,2} = u_{4,1} + u_{6,1} - u_{5,0} = 0.115+0.115 - 0.125 = 0.105$

Putting j = 2

$u_{i,3} = u_{i-1,2} + u_{i+1,2} - u_{i,1}$

i = 1, $u_{1,3} = u_{0,2} + u_{2,2} - u_{1.1}$

$u_{1,3} = 0.020$, $u_{2,3} = 0.040$, $u_{3,3} = 0.060$, $u_{4,3} = 0.075$

$u_{5,3} = 0.080$,

$u_{1,4} = 0.010$, $u_{2,4} = 0.02$, $u_{3,4} = 0.030$, $u_{4,4} = 0.040$

$u_{5,4} = 0.048$

Table 3.1 Stability of Numerical solution for wave equation

		0	0.1	0.2	0.3	0.4	0.5	0.6
	i / j	0	1	2	3	4	5	6
0	0	0	0.045	0.080	0.105	0420	0.125	0.160
0.1	1	0	0.040	0.075	0.100	0.115	0.120	0.115
0.2	2	0	0.030	0.060	0.085	0.100	0.105	0.100
0.3	3	0	0.020	0.040	0.066	0.075	0.080	0.075
0.4	4	0	0.010	0.020	0.030	0.040	0.048	0.040

Conclusion

The condition for stability was developed in accordance to the Von-Neumann condition for stability. It is also seen that the campuational results in Table 3.1 are also stable and this is the bane for this paper.

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