Journal of the Nigerian Association of Mathematical Physics
Volume **22 (November, 2012), pp 373 – 376**

Stabilization of product of eigen-values from difference scheme for the solution of hyperbolic equation

Odio Augustine Onyejuwa

Department of Mathematics,
University of Nigeria,
Nsukka, Nigeria

Abstract

This paper is on stabilization of product of eigen-values from difference scheme for the solution of hyperbolic equation. We consider a suitable difference equation usually called a leap – frog scheme given as, $U_j^{n+1} - U_j^n = \alpha[U_j^n - U_j^{n+1} - U_j^{n-1} + U_{j-1}^n]$***. Here, U depends on the time and space variables and*** α ***is a real constant. U is also differentiable in its domain of definition. Because of these properties of U, U is called an admissible function and we then apply the Taylor series expansion on it. By using trial solution for the Von Neumann method for the solution of the hyperbolic equation, we obtained the amplification matrix G(Δt,k) whose product of eigen-values of the characteristics equation for G(Δt,k) is less than one. Hence, this result shows that, the difference scheme is stable.***

Keywords: Stabilization, Eigen-values, difference scheme, courant number, Von-Neumann, admissible function and symmetric matrix.

1.0 Introduction

We consider the hyperbolic problem of a vibrating string

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2} \tag{1.1}$$

in the doman [R= $\{0 \le x \le 1\}$ x [$t > 0$],
Satisfying the following initial condition

$$U(x, 0) = f_1 (t)$$
$$U_t (x, t) = f_2 (t), \text{ for } 0 \le x \le 1 \tag{1.2}$$

and boundary conditions

$$U (0, t) = g_1 (t)$$
$$U (1,t) = g_2 (t) \text{ for all } t > 0 \tag{1.3}$$

Where c is the speed of wave and c^2 is chosen equal one here and henceforth (see Jain [1] and Tejumola [2]).
We also consider the difference scheme employed in Reference [3] commonly known as the leap-frog scheme and given by

$$U_j^{n+1} - U_j^n = \alpha[U_j^n - U_j^{n+1} - U_j^{n-1} + U_{j-1}^n] \tag{1.4}$$

where U_j^{n+1}, U_j^{n-1} and U_{j-1}^n are admissible function which take values in the space of definition [4] and by applying the Taylor series expansion [5] about (x_m, t_n) on the function $U(x_m, t_n)$, equation (1.1) is satisfied. Equation (1.4) is usually called the leap-frog scheme and written explicitly as

Corresponding author: E-mail: augustine.odio@yahoo.com, Tel. +234 8062976038

$$\frac{U_j^{n+1}-U_j^{n-1}}{2\Delta t}=-\frac{c(U_{j+1}^n-U_{j-1}^n)}{2\Delta x} \qquad (1.5)$$

$$U_j^{n+1}-U_j^{n-1}=\frac{c\Delta t}{\Delta x}(U_{j+1}^n-U_{j-1}^n$$

The quantity $\frac{c\Delta t}{\Delta x}$ is called the courant number [6], where c is wave speed, Δt is change in time and Δx is change in space. We employed a practical result due to Von Neumann for this problem. The result inform of a proposition is used to show that the result obtain for the solution of wave equation is stable.

Proposition: (Von Neumann) If $\lambda(\Delta t,k)$ is an eigenvalue of the amplification matrix $G(\Delta t,k)$ of a difference scheme, then the necessary and sufficient condition for stability are

i. $|\lambda| \leq 0(\Delta t)$

ii. $G(\Delta t.k)$ is a symmetric matrix

iii. The scheme involves only one depended variable.

2. Main Result

We consider the difference scheme for the parabolic equation as

$$U_j^{n+1}-U_j^h=\alpha[U_j^n-U_j^{n+1}-U_j^{n-1}+U_{j-i}^n] \qquad (2.1)$$

We put $V_j^{n+1}=U_j^{n-1}$

Therefore, $U_j^{n+1}-U_j^n=\alpha U_{j+1}^n-\alpha U^{n+1}{}_j-\alpha V_j^n+\alpha U_{j-1}^n$ (2.2)

$$U_j^{n+1}+\alpha U_j^{n+1} = \alpha U_{j+1}^n+U_j^n-\alpha V_j^n+\alpha U_{j-1}^n \qquad (2.3)$$

$$(1+\alpha)\ U_j^{n+1} = (U_{j+1}^n+U_{j-1}^n)=\alpha V_j^n+U_j^n \qquad (2.4)$$

Let

$$U_j^n=U_j^n e^{k\Delta x}$$

$$U_{j+1}^n=U_j^n e^{ik(x+\Delta x)} \qquad (2.5)$$

be the trial solution for the Von Neumann method for the solution of hyperbolic equation (1.1) so that

$$(1+\alpha)\ U_j^{n+1}=\alpha(e^{ikx}+e^{-ik\Delta x})U_j^n-\alpha V_j^n+U_j^n \qquad (2.6)$$

On solving equation (2.6) we obtain

$$(1+\alpha)U_j^{n+1} = 2\alpha\cos k\Delta x\, U_j^n - \alpha V_j^n + U_j^n$$

$$= 2\alpha x U_j^n - V_j^{n+1} + U_j^n$$

where s = $\cos k\Delta x$

$$(1+\alpha)U_j^{n+1} = 2\alpha s U_j^n \qquad (2.7)$$

In matrix form we have

$$\begin{pmatrix} 1+\alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{bmatrix} U_j^{n+1} \\ V_j^{n+1} \end{bmatrix} = \begin{pmatrix} 1+2\alpha s & \alpha \\ 1 & 0 \end{pmatrix}\begin{bmatrix} U_j^n \\ V_j^n \end{bmatrix} \qquad (2.8)$$

$$\begin{bmatrix} U_j^{n+1} \\ V_j^{n+1} \end{bmatrix} = \begin{pmatrix} 1+\alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{bmatrix} U_j^n \\ V_j^n \end{bmatrix} \qquad (2.9)$$

$$= \begin{pmatrix} \frac{1+2\alpha s}{1+\alpha} & \frac{-\alpha}{1+\alpha} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} U_j^n \\ V_j^n \end{pmatrix} \qquad (2.10)$$

The quantity $\begin{pmatrix} \dfrac{1+2\alpha s}{1+\alpha} & \dfrac{-\alpha}{1+\alpha} \\ 0 & 0 \end{pmatrix}$ is called

the amplification matrix and is denoted by G(Δt,k). The quantity G(Δt,k) is symmetric and hence, the matrix G(Δt,k) is invertible.
We then write equation (2.10) as

$$G\,(\Delta t,k)\begin{bmatrix} U_j^n \\ V_j^n \end{bmatrix}$$

The characteristic equation for G (Δt,k) is

$$|G| - \lambda I| = \begin{vmatrix} \dfrac{1+2\alpha s}{1+\alpha} - \lambda & \dfrac{-\alpha}{1+\alpha} \\ 1 & -\lambda \end{vmatrix} = 0 \qquad (2.11)$$

where I is the identity matrix. Now (2.11) becomes

$$= \lambda^2 - \lambda\left(\frac{1+2\alpha s}{1+\alpha} + \frac{-\alpha}{1+\alpha}\right) = 0 \qquad (2.12)$$

For stability

$$|\lambda| \le 1$$

It follows that, for the equation

$$ax^2+bx+c = 0 \qquad (2.13)$$

(i) $|b| \le 1$

(ii) $|c| \le 1$

Considering (2.12) and (2.13)

$$|\lambda_1\lambda_2| = \left|\frac{\alpha s}{|1+\alpha|}\right| < 1 \text{ for all } \alpha \qquad (2.14)$$

$$\left|\frac{\frac{1}{2}+\alpha s}{1+\alpha}\right| \le \left|\frac{\frac{1}{2}+\alpha s}{1+\alpha}\right| < 1 \text{ for all } \alpha \qquad (2.15)$$

where $|\lambda_1.\lambda_2|$ is the product of eigen–values of (2.14)

Conclusion

The result obtained in equation (2.14) shows that, the product of the eigen values λ_1 and λ_2 is less than one. This is true because the amplification matrix obtained is symmetric. Because of these reasons the amplification matrix obtained is said to be stabilized in accordance to the Von-Neumann condition for stability.

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