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An EOQ Model for Items That Exhibit Delay in Deterioration With Weibull Distribution Deterioration

[1]Yusuf A and [2]Sani B

[1]Department of Mathematics,
Kano University of Science and Technology, Wudil, Nigeria
[2]Department of Mathematics,
Ahmadu Bello University, Zaria, Nigeria

Abstract

This paper presents an EOQ inventory model for items that exhibit delay in deterioration with constant demand. A three parameter Weibull distribution is assumed as the distribution for deterioration. Numerical examples are given to illustrate the application of the model.

Keywords: Kronecker product, braid group, Burau representation, irreducible.

1.0 Introduction

Generally, inventory depletion is considered to be as a result of demand. However, the effect of deterioration of physical goods cannot be disregarded in many inventory systems. Deterioration is defined as decay, damage, spoilage or obsolescence. Food items, drugs, chemicals, electronic components and radioactive substances are some of the items in which sufficient deterioration may occur during the normal storage period. Some technical equipment such as in the manufacturing industries may suffer deterioration due to obsolescence while fashion goods may also suffer deterioration by being out of use.

The decaying inventory system was first analyzed by Ghare and Shrader [1] who developed an EOQ model with constant rate of deterioration. Covert and Philip [2] extended Ghare and Shrader's model and obtained an EOQ model for a variable rate of decay by assuming a 2-parameter Weibull distribution deterioration. Mishra [3] developed a deterioration model with finite replenishment rate. Shah and Jaiswal[4] generalized the work of Ghare and Shrader [1] to allow for backordering. Goyal [5] developed mathematical models for obtaining the economic order quantity under conditions of permissible delay in payments. Aggarwal and Jaggi [6] constructed an inventory model to determine the optimum order quantity for deteriorating items under permissible delay in payments. Jalan*et al*. [7] developed an inventory model for deteriorating items with stock-dependent demand rate. Datta and Pal [8] constructed a model on the order level inventory system with power demand pattern for items with variable rate of deterioration. Hollier and Mark [9] developed a model for inventory replenishment policies for deteriorating items in a declining market. The outline of literature on deterioration inventory can also be found in review articles by Nahmias [10], Raafat [11] and Goyal and Giri [12].

The non-instantaneous deterioration (delay in deterioration) is a situation where items do not start deteriorating immediately they are stocked. During this period, before deterioration sets in, depletion of inventory is dependent on demand only. As deterioration sets-in depletion is then dependent on both demand and deterioration. The items that exhibit delay in deterioration include farm produce such as fruits, potatoes etc. or even fashion goods such as cars, fabrics etc. Ouyang*et al.* [13] developed a model for non-instantaneous deteriorating items under permissible delay in payment. Chung [14] developed a complete proof on the solution procedure for non-instantaneous deteriorating items with permissible delay in payments. Musa and Sani [15] developed an EOQ model for items that exhibit delay in deterioration. Musa and Sani [16] also developed an EOQ model for items that exhibit delay in deterioration under permissible delay in payment. Their model is a generalization of Ouyang*et al.* [13].

[1]Corresponding author: ***Yusuf A,*** E-mail: -, Tel. +234 8036929845

Berrotoni [17] observed, while discussing the difficulties of fitting empirical data to mathematical distributions, that both leakage failure of dry batteries and life expectancy of ethical drugs could be expressed in terms of Weibull distribution. In both cases, the rate of deterioration increased with age, i.e. the longer the items remained unused, the higher the rate at which they failed. At some point of time, all units that had not been used would have failed. Perhaps the work of Berrotoni [17] prompted Covert and Philip[2] to develop an inventory model for deteriorating items with variable rate of deterioration where they used the 2-parameter Weibull distribution to represent the distribution of deterioration.

Chakrabarty*et al.* [18] developed an EOQ model for items with Weibull distribution deterioration, shortages and trended demand. Giri*et al.* [19] developed an EOQ model with Weibull deterioration distribution, shortages and ramp-type demand.

In this paper we present an EOQ model for items that exhibit delay in deterioration while considering the deterioration rate to be a 3-parameter Weibull distribution and a constant demand rate.

2.0 The Weibull Distribution

In probability theory, the Weibull distribution is a continuous probability distribution which is named after Waloddi Weibull, who described it in detail in 1951, although it was first identified by Frechet in 1927, and first applied by Rosin and Rammler in 1933 to describe the size distribution of particles and first applied by Covert and Philip in 1973 to the inventory management.

The probability density function of the general Weibull distribution (the 3-parameter Weibull distribution) is given by;

$$f(t) = \alpha\beta(t-\mu)^{(\beta-1)} e^{-\alpha(t-\mu)^{\beta}} \qquad \text{(i)}$$

Where β>0 is the shape parameter (which affect the shape of the distribution), α>0 is the scale parameter (which determines the statistical dispersion of the probability distribution), and μ>0 is the location parameter (which determines the shift of the distribution, i.e. it determines where the origin will be located). The case where μ=0 is called the 2-parameter Weibull distribution which is given by;

$$f(t) = \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}} \qquad \text{(ii)}$$

The cumulative distribution function for the 3-parameter Weibull distribution is given by;

$$F(t) = 1 - e^{-\alpha(t-\mu)^{\beta}} \qquad \text{(iii)}$$

The cumulative distribution function of the 2-parameter Weibull distribution is given by;

$$F(t) = 1 - e^{-\alpha t^{\beta}} \qquad \text{(iv)}$$

3.0 Justification for Using the Weibull Distribution to Represent The Time To Deteriorate

The instantaneous rate of deterioration of the non-deteriorated inventory $\varphi(t)$, at time *t*, can be obtained from;

$$\varphi(t) = \frac{f(t)}{1-F(t)} \qquad \text{(v)}$$

Where *f(t)* is the probability density function of the Weibull distribution and *F(t)* is the cumulative distribution function. Substituting *f(t)* and *F(t)* from (ii) and (iv) into (v) and simplifying we obtain the rate of deterioration for the 2-parameter Weibull distribution as follows;

$$\varphi(t) = \alpha\beta t^{\beta-1} \qquad \text{(vi)}$$

Also, substituting *f(t)* and *F(t)* from (i) and (iii) into (v) and simplifying we obtain the rate of deterioration for the 3-parameter Weibull distribution as follows;

$$\varphi(t) = \alpha\beta(t-\mu)^{\beta-1} \qquad \text{(vii)}$$

The rate of deterioration-time relationship for the 2-parameter Weibull distribution is as shown in Fig.1 and it can be seen from Fig.1 that the 2-parameter Weibull distribution is appropriate for items with increasing rate of deterioration only if the initial rate is approximately zero or decreasing rate of deterioration only if the initial rate of deterioration is extremely high, (Chakrabarty*et al.* [18]).

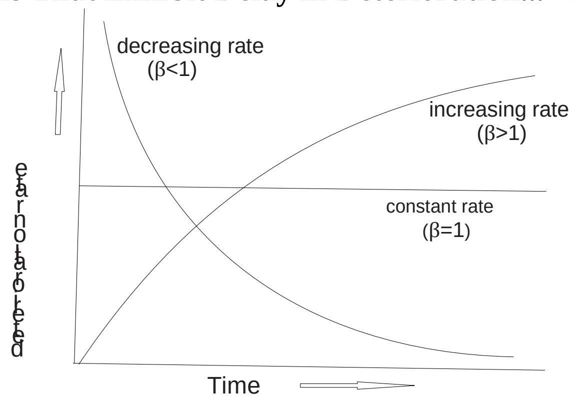

Fig. 1 Rate of deterioration-time relationship for a 2-parameter Weibull distribution

To remove these limitations, Philip [20] developed a generalized EOQ model with the 3-parameter Weibull distribution to represent the time of deterioration. The rate of deterioration-time relationship for the 3-parameter Weibull distribution is depicted in Fig. 2.

It is clear from Fig. 2 that the 3-parameter Weibull distribution is suitable for items with any initial value of the rate of deterioration and also for items which start deteriorating only after a certain period of time, (Chakrabarty*et al.* [18]).

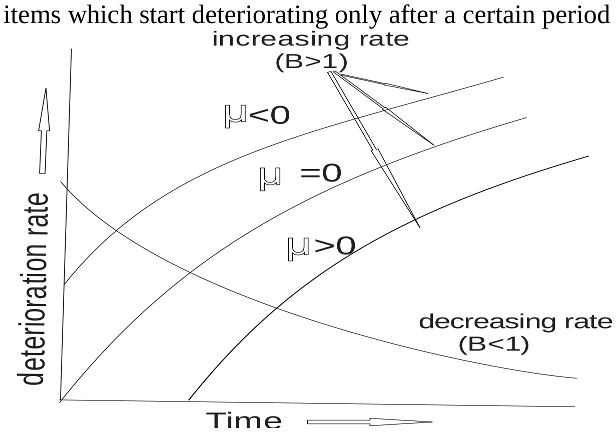

Fig. 2 Rate of deterioration-time relationship for a 3-parameter Weibull distribution

Notation and Assumptions

Notation

D_1	Demand rate (units per unit time) during the period before deterioration sets in
D_2	Demand rate (units per unit time) after deterioration sets in
Q	Order quantity (units per order)
T	Inventory cycle length (time units)
c	Unit cost of the item (in Naira)
A	Ordering cost per order (Naira per Order)
i	Inventory carrying charge (excluding interest charges, Naira per unit time)
$\varphi(t)$	Rate of deterioration
I_0	Initial Inventory level
I_d	inventory level at the time deterioration sets in
$I_d(t)$	Inventory level at any time t when deterioration sets in
T_1	Time deterioration sets in
T_2	Difference between the cycle length and time deterioration sets in
d_T	Number of items that deteriorate within the time period $[T_1,T]$

Assumptions

(i) Instantaneous replenishment
(ii) Unconstrained suppliers capital
(iii) Shortages not allowed
(iv) Negligible lead time
(v) Three-parameter Weibull distribution deterioration given by

$$\varphi(t) = \alpha\beta(t-\mu)^{\beta-1};\ t>0 \text{ and } t\geq\mu$$

Where $\alpha>0$, $\beta>0$ and $\mu>0$ are the scale parameter, shape parameter and location parameter respectively. Note that if $t<\mu$, then for some values of β, $\varphi(t)<0$.

4.0 The Mathematical Model

Let $I(t)$ be the inventory level at any time t, in the region $[0,T_1]$. Depletion of inventory from the beginning of the cycle up to the time deterioration sets in will occur only due to demand. Within the region $[T_1,T]$depletion of the inventory will depend on both demand and deterioration. Fig. 3 gives a graphical representation of the situation.

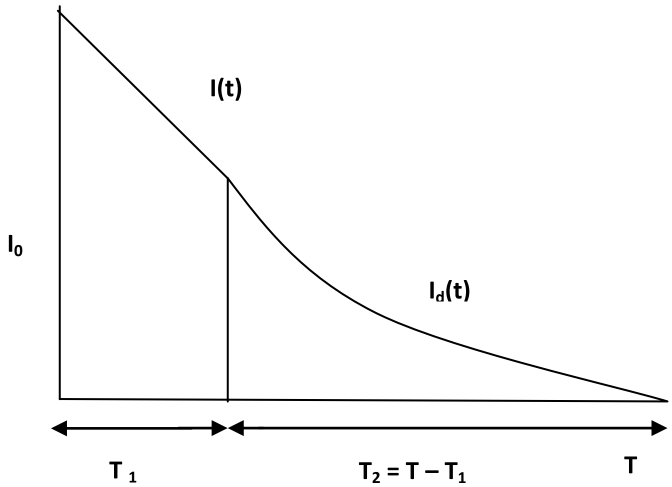

Fig. 3 Inventory movement in a delayed deterioration situation

The differential equation that describes the state of the inventory level, $I(t)$, in the interval $[0,T_1]$ is given by;

$$\frac{dI(t)}{dt}=-D_1, \tag{1}$$

The solution of (1) is given by;

$$I(t)=-D_1t+k_1 \tag{2}$$

Where k_1 is an arbitrary constant

Applying the boundary condition $I(0)=I_o$ to (2) we have $k_1=I_o$

$$\therefore\ I(t)=-D_1t+I_o \tag{3}$$

Also, applying the boundary condition $I(T_1)=I_d$ to (2) we have

$$I(t)=-D_1t+D_1T_1+I_d$$

$$\Rightarrow\ I(t)=I_d+(T_1-t)D_1 \tag{4}$$

The differential equation that describes the inventory level within the region $[T_1,T]$ is given by;

$$\frac{dI_d(t)}{dt}+\varphi(t)I_d(t)=-D_2;\quad T_1\le t\le T \qquad \text{where } \varphi(t) \text{ is as defined in (vii)}$$

$$\Rightarrow\frac{dI_d(t)}{dt}+\alpha\beta(t-\mu)^{\beta-1}I_d(t)=-D_2; \tag{5}$$

Solving (5) using integrating factor method, we have;

$$\frac{dI_d(t)}{dt}e^{\alpha(t-\mu)^\beta}+\alpha\beta(t-\mu)^{\beta-1}I_d(t)e^{\alpha(t-\mu)^\beta}=-D_2e^{\alpha(t-\mu)^\beta}$$

$$\Rightarrow\quad \frac{d\left(I_d(t)e^{\alpha(t-\mu)^\beta}\right)}{dt}=-D_2e^{\alpha(t-\mu)^\beta}$$

$$\Rightarrow\quad I_d(t)e^{\alpha(t-\mu)^\beta}=-D_2\int e^{\alpha(t-\mu)^\beta} \tag{6}$$

To find $\int e^{\alpha(t-\mu)^\beta}dt$ let $\gamma=\alpha(t-\mu)^\beta \ \Rightarrow\ d\gamma=\alpha\beta(t-\mu)^{\beta-1}dt$

$$\Rightarrow\ dt = \frac{1}{\alpha\beta(t-\mu)^{\beta-1}}d\gamma \quad where\ \ t = \left(\frac{\gamma}{\alpha}\right)^{1/\beta} + \mu$$

$$\Rightarrow\ dt = \frac{1}{\alpha^{1/\beta}\beta\gamma^{\frac{\beta-1}{\beta}}}d\gamma \quad \Rightarrow dt = \frac{1}{\alpha^{1/\beta}\beta}\gamma^{\left(\frac{1}{\beta}\right)-1}d\gamma$$

$$\therefore\ \int e^{\alpha(t-\mu)^{\beta}}dt = \frac{1}{\alpha^{1/\beta}\beta}\int\gamma^{\left(\frac{1}{\beta}\right)-1}e^{\gamma}d\gamma, \quad where\ \ \gamma = \alpha(t-\mu)^{\beta}$$

Expanding e^{γ} using Maclaurine series we have;

$$e^{\gamma} = 1 + \gamma + \frac{\gamma^2}{2!} + \frac{\gamma^3}{3!} + \frac{\gamma^4}{4!} + ---$$

$$\Rightarrow \int e^{\alpha(t-\mu)^{\beta}}dt = \frac{1}{\alpha^{1/\beta}\beta}\int\gamma^{\left(\frac{1}{\beta}\right)-1}\left[1 + \gamma + \frac{\gamma^2}{2!} + \frac{\gamma^3}{3!} + \frac{\gamma^4}{4!} + ---\right]d\gamma$$

$$\Rightarrow \int e^{\alpha(t-\mu)^{\beta}}dt = \frac{1}{\alpha^{1/\beta}\beta}\int\left[\gamma^{\left(\frac{1}{\beta}\right)-1} + \gamma^{\left(\frac{1}{\beta}\right)} + \frac{\gamma^{\left(\frac{1}{\beta}\right)+1}}{2!} + \frac{\gamma^{\left(\frac{1}{\beta}\right)+2}}{3!} + \frac{\gamma^{\left(\frac{1}{\beta}\right)+3}}{3!} + \frac{\gamma^{\left(\frac{1}{\beta}\right)+4}}{4!} + ---\right]d\gamma$$

$$\Rightarrow\ \int e^{\alpha(t-\mu)^{\beta}}dt = \frac{1}{\alpha^{1/\beta}\beta}\left[\frac{\gamma^{\left(\frac{1}{\beta}\right)}}{\left(\frac{1}{\beta}\right)} + \frac{\gamma^{\left(\frac{1}{\beta}\right)+1}}{\left(\frac{1}{\beta}\right)+1} + \frac{\gamma^{\left(\frac{1}{\beta}\right)+2}}{2!\left(\left(\frac{1}{\beta}\right)+2\right)} + \frac{\gamma^{\left(\frac{1}{\beta}\right)+3}}{3!\left(\left(\frac{1}{\beta}\right)+3\right)} + \frac{\gamma^{\left(\frac{1}{\beta}\right)+4}}{4!\left(\left(\frac{1}{\beta}\right)+4\right)} + ---\right] + k_2$$

Where k_2 is an arbitrary constant

$$\Rightarrow \int e^{\alpha(t-\mu)^{\beta}}dt = \frac{1}{\alpha^{1/\beta}}\gamma^{\frac{1}{\beta}}\left[1 + \frac{\gamma}{1+\beta} + \frac{\gamma^2}{2!(1+2\beta)} + \frac{\gamma^3}{3!(1+3\beta)} + \frac{\gamma^4}{4!(1+4\beta)} + ---\right] + k_2$$

$$\Rightarrow \int e^{\alpha(t-\mu)^{\beta}}dt = \frac{1}{\alpha^{1/\beta}}\gamma^{\frac{1}{\beta}}\sum_{n=0}^{\infty}\left[\frac{\gamma^n}{n!(1+n\beta)}\right] + k_2, \quad \text{Where } n \text{ is an integer} \tag{7}$$

$$\Rightarrow \int e^{\alpha(t-\mu)^{\beta}}dt = \frac{1}{\alpha^{1/\beta}}\left[\alpha(t-\mu)^{\beta}\right]^{\frac{1}{\beta}}\sum_{n=0}^{\infty}\left[\frac{\left[\alpha(t-\mu)^{\beta}\right]^n}{n!(1+n\beta)}\right] + k_2$$

$$\Rightarrow \int e^{\alpha(t-\mu)^{\beta}}dt = \sum_{n=0}^{\infty}\left[\frac{\alpha^n(t-\mu)^{\beta n+1}}{n!(1+n\beta)}\right] + k_2 \tag{8}$$

Substituting (8) into (6) we have;

$$\Rightarrow I_d(t)e^{\alpha(t-\mu)^{\beta}} = -D_2\sum_{n=0}^{\infty}\left[\frac{\alpha^n(t-\mu)^{\beta n+1}}{n!(1+n\beta)}\right] + k_2$$

$$\Rightarrow\ I_d(t) = -D_2e^{-\alpha(t-\mu)^{\beta}}\sum_{n=0}^{\infty}\left[\frac{\alpha^n(t-\mu)^{1+n\beta}}{n!(1+n\beta)}\right] + k_2e^{-\alpha(t-\mu)^{\beta}} \tag{9}$$

Substituting boundary condition $I(T_1)=I_d$ we have;

$$-D_2e^{-\alpha(T_1-\mu)^{\beta}}\sum_{n=0}^{\infty}\left[\frac{\alpha^n(T_1-\mu)^{1+n\beta}}{n!(1+n\beta)}\right] + k_2e^{-\alpha(T_1-\mu)^{\beta}} = I_d$$

$$\Rightarrow k_2 = I_d e^{\alpha(T_1-\mu)^\beta} + D_2 \sum_{n=0}^{\infty}\left[\frac{\alpha^n (T_1-\mu)^{1+n\beta}}{n!(1+n\beta)}\right] \tag{10}$$

Substituting (10) into (9) we have;

$$\Rightarrow\ I_d(t) = -D_2 e^{-\alpha(t-\mu)^\beta}\sum_{n=0}^{\infty}\left[\frac{\alpha^n (t-\mu)^{1+n\beta}}{n!(1+n\beta)}\right] + \left(I_d e^{\alpha(T_1-\mu)^\beta} + D_2\sum_{n=0}^{\infty}\left[\frac{\alpha^n (T_1-\mu)^{1+n\beta}}{n!(1+n\beta)}\right]\right)e^{-\alpha(t-\mu)^\beta}$$

$$\therefore I_d(t) = I_d e^{\alpha(T_1-\mu)^\beta} e^{-\alpha(t-\mu)^\beta} - D_2 e^{-\alpha(t-\mu)^\beta}\sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\left[(t-\mu)^{1+n\beta} - (T_1-\mu)^{1+n\beta}\right]\right] \tag{11}$$

Substituting the boundary condition $I_d(T)=0$ we have;

$$I_d e^{\alpha(T_1-\mu)^\beta} e^{-\alpha(T-\mu)^\beta} - D_2 e^{-\alpha(T-\mu)^\beta}\sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\left[(T-\mu)^{1+n\beta} - (T_1-\mu)^{1+n\beta}\right]\right] = 0$$

$$\Rightarrow I_d = D_2 e^{-\alpha(T_1-\mu)^\beta}\sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\left[(T-\mu)^{1+n\beta} - (T_1-\mu)^{1+n\beta}\right]\right] \tag{12}$$

Substituting (12) into (11) we have;

$$I_d(t) = D_2 e^{-\alpha(T_1-\mu)^\beta}\sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\left[(T-\mu)^{1+n\beta} - (T_1-\mu)^{1+n\beta}\right]\right] e^{\alpha(T_1-\mu)^\beta} e^{-\alpha(t-\mu)^\beta}$$
$$- D_2 e^{-\alpha(t-\mu)^\beta}\sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\left[(t-\mu)^{1+n\beta} - (T_1-\mu)^{1+n\beta}\right]\right]$$

$$\Rightarrow\ I_d(t) = D_2 e^{-\alpha(t-\mu)^\beta}\sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\left[(T-\mu)^{1+n\beta} - (T_1-\mu)^{1+n\beta}\right]\right]$$
$$- D_2 e^{-\alpha(t-\mu)^\beta}\sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\left[(t-\mu)^{1+n\beta} - (T_1-\mu)^{1+n\beta}\right]\right]$$

$$\Rightarrow I_d(t) = D_2 e^{-\alpha(t-\mu)^\beta}\sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\left[(T-\mu)^{1+n\beta} - (T_1-\mu)^{1+n\beta} - (t-\mu)^{1+n\beta} + (T_1-\mu)^{1+n\beta}\right]\right]$$

$$\therefore\ I_d(t) = D_2 e^{-\alpha(t-\mu)^\beta}\sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\left[(T-\mu)^{1+n\beta} - (t-\mu)^{1+n\beta}\right]\right] \tag{13}$$

Substituting (12) into (4) we have;

$$I(t) = D_2 e^{-\alpha(T_1-\mu)^\beta}\sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\left[(T-\mu)^{1+n\beta} - (T_1-\mu)^{1+n\beta}\right]\right] + (T_1-t)D_1 \tag{14}$$

Total demand within the region $T_1 \le t \le T$ is given by D_2T_2

Therefore, the number of items that deteriorate within $T_1 \le t \le T$ is given by;

$$d_T = I_d - D_2T_2 = I_d - D_2(T-T_1)$$

$$\Rightarrow\ d_T = D_2 e^{-\alpha(T_1-\mu)^\beta}\sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\left[(T-\mu)^{1+n\beta} - (T_1-\mu)^{1+n\beta}\right]\right] - D_2(T-T_1) \tag{15}$$

We let the holding cost be C_H, where;

$$C_H = ic\left[\int_0^{T_1} I(t)dt + \int_{T_1}^{T} I_d(t)dt\right] \tag{16}$$

$$\int_0^{T_1} I(t)dt = \int_0^{T_1}\left(D_2 e^{-\alpha(T_1-\mu)^\beta} \sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\left[(T-\mu)^{1+n\beta} - (T_1-\mu)^{1+n\beta}\right]\right] + (T_1 - t)D_1 \right) dt$$

$$\Rightarrow \int_0^{T_1} I(t)dt = D_2 T_1 e^{-\alpha(T_1-\mu)^\beta} \sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\left[(T-\mu)^{1+n\beta} - (T_1-\mu)^{1+n\beta}\right]\right] + \frac{D_1 T_1^{\,2}}{2} \tag{17}$$

$$and \int_{T_1}^{T} I_d(t)dt = \int_{T_1}^{T}\left(D_2 e^{-\alpha(t-\mu)^\beta} \sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\left[(T-\mu)^{1+n\beta} - (t-\mu)^{1+n\beta}\right]\right]\right) dt$$

$$= \int_{T_1}^{T} D_2 e^{-\alpha(t-\mu)^\beta} \sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}(T-\mu)^{1+n\beta}\right] dt - \int_{T_1}^{T} D_2 e^{-\alpha(t-\mu)^\beta} \sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}(t-\mu)^{1+n\beta}\right] dt$$

$$= \sum_{n=0}^{\infty}\left[\int_{T_1}^{T} D_2 e^{-\alpha(t-\mu)^\beta}\left(\frac{\alpha^n}{n!(1+n\beta)}(T-\mu)^{1+n\beta}\right) dt\right] - \sum_{n=0}^{\infty}\left[\int_{T_1}^{T} D_2 e^{-\alpha(t-\mu)^\beta}\left(\frac{\alpha^n}{n!(1+n\beta)}(t-\mu)^{1+n\beta}\right) dt\right]$$

$$= D_2 \sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}(T-\mu)^{1+n\beta}\int_{T_1}^{T} e^{-\alpha(t-\mu)^\beta} dt\right] - D_2 \sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\int_{T_1}^{T} e^{-\alpha(t-\mu)^\beta}(t-\mu)^{1+n\beta} dt\right] \tag{18}$$

To find $\int_{T_1}^{T} e^{-\alpha(t-\mu)^\beta} dt$ let $\gamma = -\alpha(t-\mu)^\beta \Rightarrow d\gamma = -\alpha\beta(t-\mu)^{\beta-1} dt$

$$\Rightarrow dt = -\frac{1}{\alpha\beta(t-\mu)^{\beta-1}} d\gamma \quad where \quad t = \left(-\frac{\gamma}{\alpha}\right)^{1/\beta} + \mu$$

$$\Rightarrow dt = \frac{1}{(-\alpha)^{1/\beta}\beta\gamma^{\frac{\beta-1}{\beta}}} d\gamma \quad \Rightarrow dt = -\frac{1}{(-\alpha)^{1/\beta}\beta}\gamma^{\left(\frac{1}{\beta}\right)-1} d\gamma$$

$$\therefore \int_{T_1}^{T} e^{-\alpha(t-\mu)^\beta} dt = \frac{1}{(-\alpha)^{1/\beta}\beta}\int_{T_1}^{T} \gamma^{\left(\frac{1}{\beta}\right)-1} e^{\gamma} d\gamma \quad where \quad \gamma = -\alpha(t-\mu)^\beta \tag{19}$$

Using Maclaurine series expansion of e^{γ} in (19) and simplifying as was done in deriving equation (7) we have;

$$\Rightarrow \int_{T_1}^{T} e^{-\alpha(t-\mu)^\beta} dt = \left| \frac{1}{(-\alpha)^{1/\beta}} \gamma^{\frac{1}{\beta}} \sum_{m=0}^{\infty}\left[\frac{\gamma^m}{m!(1+m\beta)}\right]\right|_{T_1}^{T}$$

$$\Rightarrow \int_{T_1}^{T} e^{-\alpha(t-\mu)^\beta} dt = \left| \frac{1}{(-\alpha)^{1/\beta}} \left[-\alpha(t-\mu)^\beta\right]^{\frac{1}{\beta}} \sum_{m=0}^{\infty}\left[\frac{\left[-\alpha(t-\mu)^\beta\right]^m}{m!(1+m\beta)}\right]\right|_{T_1}^{T}$$

$$\Rightarrow \int_{T_1}^{T} e^{-\alpha(t-\mu)^\beta} dt = \sum_{m=0}^{\infty}\left[\frac{(-\alpha)^m}{m!(1+m\beta)}\left[(T-\mu)^{1+\beta m} - (T_1-\mu)^{1+\beta m}\right]\right] \tag{20}$$

To find $\int_{T_1}^{T} e^{\alpha(t-\mu)^\beta}(t-\mu)^{1+n\beta})dt$,we use the Maclaurine series of $e^{\alpha(t-\mu)^\beta}$ which is given by

$$e^{\alpha(t-\mu)^\beta} = 1 + \alpha(t-\mu)^\beta + \frac{\alpha^2(t-\mu)^{2\beta}}{2!} + \frac{\alpha^3(t-\mu)^{3\beta}}{3!} + \frac{\alpha^4(t-\mu)^{4\beta}}{4!} + ---$$

$$\Rightarrow \int_{T_1}^{T} e^{\alpha(t-\mu)^\beta}(t-\mu)^{1+n\beta})dt = \int_{T_1}^{T}\left((t-\mu)^{1+n\beta} + \alpha(t-\mu)^{1+\beta(n+1)} + \frac{\alpha^2(t-\mu)^{1+\beta(n+2)}}{2!} + \frac{\alpha^3(t-\mu)^{1+\beta(n+3)}}{3!} + --- \right) dt$$

$$\Rightarrow \int_{T_1}^{T} e^{\alpha(t-\mu)^{\beta}} (t-\mu)^{1+n\beta})dt = \left(\frac{(T-\mu)^{2+\beta n}}{2+\beta n} + \alpha \frac{(T-\mu)^{2+\beta(n+1)}}{2+\beta(n+1)} + \frac{\alpha^2 (T-\mu)^{2+\beta(n+2)}}{2!(2+\beta(n+2))} + \frac{\alpha^3 (T-\mu)^{2+\beta(n+3)}}{3!(2+\beta(n+3))} + --- \right)$$

$$- \left(\frac{(T_1-\mu)^{2+\beta n}}{2+\beta n} + \alpha \frac{(T_1-\mu)^{2+\beta(n+1)}}{2+\beta(n+1)} + \frac{\alpha^2 (T_1-\mu)^{2+\beta(n+2)}}{2!(2+\beta(n+2))} + \frac{\alpha^3 (T_1-\mu)^{2+\beta(n+3)}}{3!(2+\beta(n+3))} + --- \right)$$

$$\Rightarrow \int_{T_1}^{T} e^{\alpha(t-\mu)^{\beta}} (t-\mu)^{1+n\beta})dt = \sum_{m=0}^{\infty} \left[\frac{\alpha^m}{m!(2+\beta(n+m))} \left[(T-\mu)^{2+\beta(n+m)} - (T_1-\mu)^{2+\beta(n+m)} \right] \right] \quad (21)$$

Substituting (21) and (20) into (18) we have

$$\therefore \int_{T_1}^{T} I_d(t)dt = D_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\frac{\alpha^n}{n!(1+n\beta)} \left(\frac{(-\alpha)^m}{m!(1+m\beta)} \left((T-\mu)^{2+\beta(n+m)} - (T-\mu)^{1+n\beta} (T_1-\mu)^{1+m\beta} \right) \right) \right]$$

$$- D_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\frac{\alpha^n}{n!(1+n\beta)} \left(\frac{\alpha^m}{m!(2+\beta(n+m))} \left((T-\mu)^{2+\beta(n+m)} - (T_1-\mu)^{2+\beta(n+m)} \right) \right) \right] (22)$$

Substituting (22) and (17) into (16) we have;

$$C_H = icD_2 T_1 e^{-\alpha(T_1-\mu)^{\beta}} \sum_{n=0}^{\infty} \left[\frac{\alpha^n}{n!(1+n\beta)} \left[(T-\mu)^{1+n\beta} - (T_1-\mu)^{1+n\beta} \right] \right] + \frac{icD_1 T_1^{\ 2}}{2}$$

$$+ icD_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\frac{\alpha^n}{n!(1+n\beta)} \left(\frac{(-\alpha)^m}{m!(1+m\beta)} \left((T-\mu)^{2+\beta(n+m)} - (T-\mu)^{1+n\beta} (T_1-\mu)^{1+m\beta} \right) \right) \right]$$

$$- icD_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\frac{\alpha^n}{n!(1+n\beta)} \left(\frac{\alpha^m}{m!(2+\beta(n+m))} \left((T-\mu)^{2+\beta(n+m)} - (T_1-\mu)^{2+\beta(n+m)} \right) \right) \right] \quad (23)$$

The total variable cost is given by;

$$T_{VC} = A + cd_T + C_H \quad (24)$$

$$= A + cD_2 e^{-\alpha(T_1-\mu)^{\beta}} \sum_{n=0}^{\infty} \left[\frac{\alpha^n}{n!(1+n\beta)} \left[(T-\mu)^{1+n\beta} - (T_1-\mu)^{1+n\beta} \right] \right] - cD_2 (T - T_1)$$

$$+ icD_2 T_1 e^{-\alpha(T_1-\mu)^{\beta}} \sum_{n=0}^{\infty} \left[\frac{\alpha^n}{n!(1+n\beta)} \left[(T-\mu)^{1+n\beta} - (T_1-\mu)^{1+n\beta} \right] \right] + \frac{icD_1 T_1^{\ 2}}{2}$$

$$+ icD_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\frac{\alpha^n}{n!(1+n\beta)} \left(\frac{(-\alpha)^m}{m!(1+m\beta)} \left((T-\mu)^{2+\beta(n+m)} - (T-\mu)^{1+n\beta} (T_1-\mu)^{1+m\beta} \right) \right) \right]$$

$$- icD_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\frac{\alpha^n}{n!(1+n\beta)} \left(\frac{\alpha^m}{m!(2+\beta(n+m))} \left((T-\mu)^{2+\beta(n+m)} - (T_1-\mu)^{2+\beta(n+m)} \right) \right) \right] \quad (25)$$

Since μ>0 is the location parameter (which determines the shift of the distribution, i.e. it determines where the origin will be located) then letting $\mu = T_1$ we have;

$$T_{VC} = cD_2 (1 + iT_1) \sum_{n=0}^{\infty} \left[\frac{\alpha^n}{n!(1+n\beta)} (T - T_1)^{1+n\beta} \right] + icD_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\frac{\alpha^n}{n!(1+n\beta)} \left(\frac{(-\alpha)^m}{m!(1+m\beta)} (T - T_1)^{2+\beta(n+m)} \right) \right]$$

$$- icD_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\frac{\alpha^n}{n!(1+n\beta)} \left(\frac{\alpha^m}{m!(2+\beta(n+m))} (T - T_1)^{2+\beta(n+m)} \right) \right] + A - cD_2 (T - T_1) + \frac{icD_1 T_1^{\ 2}}{2} \quad (26)$$

The total variable cost per unit time is then given by;

$$T_{VC}(T) = cD_2(1+iT_1)\sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\frac{(T-T_1)^{1+n\beta}}{T}\right] + icD_2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\left(\frac{(-\alpha)^m}{m!(1+m\beta)}\frac{(T-T_1)^{2+\beta(n+m)}}{T}\right)\right]$$

$$-icD_2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\left(\frac{\alpha^m}{m!(2+\beta(n+m))}\frac{(T-T_1)^{2+\beta(n+m)}}{T}\right)\right] + \frac{A}{T} - \frac{cD_2(T-T_1)}{T} + \frac{icD_1T_1^2}{2T} \tag{27}$$

Differentiating (27) with respect to T and equating to zero gives the value of T which minimizes the total variable cost per unit time, provided $\frac{d^2T_{vc}(T)}{dT^2} > 0$. This is given as follows;

$$\frac{dT_{VC}(T)}{dT} = cD_2(1+iT_1)\sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\frac{T(1+n\beta)(T-T_1)^{n\beta} - (T-T_1)^{1+n\beta}}{T^2}\right]$$

$$+icD_2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\frac{(-\alpha)^m}{m!(1+m\beta)}\frac{T(2+\beta(n+m))(T-T_1)^{1+\beta(n+m)} - (T-T_1)^{2+\beta(n+m)}}{T^2}\right]$$

$$-icD_2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\frac{\alpha^m}{m!(2+\beta(n+m))}\frac{T(2+\beta(n+m))(T-T_1)^{1+\beta(n+m)} - (T-T_1)^{2+\beta(n+m)}}{T^2}\right]$$

$$-\frac{A}{T^2} - \frac{cD_2T_1}{T^2} - \frac{icD_1T_1^2}{2T^2} = 0$$

$$\Rightarrow cD_2(1+iT_1)\sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}(T-T_1)^{n\beta}(T_1+n\beta T)\right]$$

$$+icD_2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\frac{(-\alpha)^m}{m!(1+m\beta)}(T-T_1)^{1+\beta(n+m)}(T+T_1+\beta(n+m)T)\right]$$

$$-icD_2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\frac{\alpha^m}{m!(2+\beta(n+m))}(T-T_1)^{1+\beta(n+m)}(T+T_1+\beta(n+m)T)\right] - A - cD_2T_1 - \frac{icD_1T_1^2}{2} = 0$$

Approximating this value by considering only the first two terms of the series, gives;

$$\approx cD_2(1+iT_1)\left[T_1 + \frac{\alpha}{1+\beta}(T-T_1)^{\beta}(T_1+\beta T)\right]$$

$$+icD_2(T-T_1)\left[\frac{1}{2}(T+T_1) - \frac{\alpha}{1+\beta}(T-T_1)^{\beta}(T+T_1+\beta T) - \frac{3}{2}\left(\frac{\alpha}{1+\beta}\right)^2(T-T_1)^{2\beta}(T+T_1+2\beta T)\right]$$

$$-\left(A + cD_2T_1 + \frac{icD_1T_1^2}{2}\right) = 0 \tag{28}$$

Equation (28) can be used to determine the best time period T which minimizes the total variable cost, and this can be reduced from (27) to;

$$T_{VC}(T) \approx \frac{cD_2}{T}(1+iT_1)(T-T_1)\left[1+\frac{\alpha}{1+\beta}(T-T_1)^{\beta}\right]$$
$$+\frac{icD_2}{T}(T-T_1)^2\left[\frac{1}{2}-\frac{\alpha}{1+\beta}(T-T_1)^{\beta}-\frac{3}{2}\left(\frac{\alpha}{1+\beta}\right)^2(T-T_1)^{2\beta}\right]+\frac{1}{T}\left(A+cD_2T_1+\frac{icD_1T_1^{\ 2}}{2}-cD_2T\right) \qquad (29)$$

The EOQ of the corresponding time period T will be determined from:

$$Q = D_1T_1 + D_2T_2 + d_T = D_1T_1 + D_2(T-T_1) + d_T$$
$$\Rightarrow Q = D_1T_1 + D_2e^{-\alpha(T_1-\mu)^{\beta}}\sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}\left[(T-\mu)^{1+n\beta}-(T_1-\mu)^{1+n\beta}\right]\right]$$
$$\Rightarrow Q = D_1T_1 + D_2\sum_{n=0}^{\infty}\left[\frac{\alpha^n}{n!(1+n\beta)}(T-T_1)^{1+n\beta}\right] \quad \text{after putting } \mu = T_1 \text{ as explained earlier}$$
$$\Rightarrow Q \approx D_1T_1 + D_2\left[(T-T_1)+\frac{\alpha}{1+\beta}(T-T_1)^{1+\beta}\right] \qquad (30)$$

After considering only the first two terms of the series, which is particularly true for *T*∈*[0, 1]*

Numerical Examples

Examples are given to determine the best cycle length. Table 1 below gives details of the examples. In all the examples, $\frac{d^2T_{VC}(T)}{dT}$ is found to be greater than zero showing that the cost function is convex in all the examples.

Table 1: Numerical Examples

S/N	*A(₦)*	*C(₦)*	*D1(units)*	*D2(units)*	*T1(days)*	*i*	α	β	*T(days)*	*TVC(₦)*	*EOQ(units)*
1	230	60	800	300	7	0.13	5	1	27	5021	34
2	200	50	500	250	14	0.15	7.2	3	90	1215	71
3	250	70	1000	500	21	0.14	1	1	47	3007	93
4	300	80	2000	500	28	0.12	8.5	2	58	2931	194
5	350	35	900	400	35	0.11	2	8	223	1070	293

5.0 Sensitivity Analysis

Sensitivity analysis depicts the extent to which the output of a model is affected by changes or errors in its input parameters. In this section, we examine the sensitivity of the scheduling period T, order quantity Q and the total variable cost $T_{VC}(T)$ of the inventory model with respect to the input parameters A, c, D_1, D_2, T_1, i, α and β.

Table 2 gives results of some sensitivity analysis carried out on the first example of table 1, which shows that the model is highly sensitive to changes in the parameter β. It also shows that the model is moderately sensitive to changes in the parameters A, c, D_1, D_2 and α (where D_1 and D_2 change concurrently), slightly sensitive to changes in T_1 and almost insensitive to changes in the parameter i.

The analysis also shows that increase in the value of β results in considerable increase in the value of the outputs T, Q and $T_{VC}(T)$. On the other hand increase in the value of α results in considerable decrease in the values of T, Q and $T_{VC}(T)$.

Thus, sufficient care should be taken to estimate the parameters α and β in using the model.

Table 2: Sensitivity Analysis

Parameter	**%** change in the parameter values	**%** change in the results		
		T	TVC	EOQ
A	-50	26	36	21
	-20	7	13	6
	20	11	12	9
	50	22	28	21
C	-50	41	24	35
	-20	11	8	9
	20	7	7	6
	50	15	16	12
D_1 and D_2	-50	41	24	32
	-20	11	8	12
	20	7	7	12
	50	14	16	32
T_1	-50	0	14	12
	-20	0	5	9
	20	4	5	9
	50	4	11	15
i	-50	0	1	0
	-20	0	0	0
	20	0	0	0
	50	0	1	0
α	-50	41	23	26
	-20	11	8	9
	20	7	7	6
	50	15	15	9
β	-50	56	99	41
	-20	22	30	15
	20	30	20	15
	50	63	40	41

6.0 Conclusion

In this paper, we present a mathematical model on the inventory of deteriorating items which do not start deteriorating immediately they are stocked. Items that have this property include farm produce, radioactive chemicals, fashion goods and so on. A 3-parameter Weibull distribution is used as the distribution for deterioration as in Chakrabarty*et al.* [18]. Numerical examples are given to determine the best cycle length and the corresponding total variable cost and order quantity. Further research can be done to determine the applicability of the scale parameter α, and the shape parameter β.

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