

Differencing When Growth Curve Is Exponential

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Abstract

In this paper, we demonstrate the difficulties in using differencing as a procedure for isolating the exponential trend curve. For the polynomial curve of order d, the dth order differencing is required to isolate the trend. For the exponential growth curve the dth order backward differencing can hardly isolate the trend unless for very high values of d. On the other hand, successive ratios produce a constant. So we recommend successive ratios as a means of isolating the exponential growth curve from a series.

Keywords: Trend component, exponential growth curve, polynomial curve, backward differencing operator, successive ratios.

1.0 Introduction

Autoregressive Integrated Moving Average models [ARIMA(p, d, q)] are, in theory, the most general class of models for forecasting a nonseasonal time series which can be stationarized by transformations such as differencing. Remember that a nonseasonal model is classified as an ARIMA(p, d, q) model, where:

- p is the number of autoregressive terms,
- d is the number of nonseasonal differences, and
- q is the number of moving average terms

To identify the appropriate ARIMA model for a time series, you begin by identifying the order, q, of differencing needed to stationarize the series [1-3].

Must we always difference a nonseasonal time series to make it stationary?. The answer is yes when the trend curve can adequately be represented by a polynomial of a finite order r. For many order trend curves, such as the exponential growth curve, the answer may be yes (when appropriate polynomial expansion is obtainable) or no (when inappropriate polynomial expansion is obtained)

What alternatives exist when differencing is not possible? We show that value relatives provide a better alternative for the exponential trend curve.

2. 0 Exponential Growth Curve and Uses

When a quantity grows or decays by a fixed percent at regular intervals, the pattern of growth or decay can be represented by the exponential function

$$X_t = b e^{ct} = b(1+r)^t \quad (1)$$

where X_t = value of the quantity at time t ;

b = initial value when t = 0 before measuring growth/decay ;

c (r) = growth / decay rate (r is often a percent); t = number of time intervals that have passed

$$e^c = (1+r) \Rightarrow c = \log_e (1+r) \quad (2)$$

Exponential growth ($r > 0 / c > 0$) and exponential decay ($r < 0 / c < 0$).

Growth curves are employed in many disciplines like studies of population, bacteria, the AIDS virus, growth of children, cancer cell growth, body weight or biomass, radioactive substances, electricity, temperatures and credit payments, to mention a few. For additional information on uses of growth curves, see http://en.wikipedia.org/wiki/Growth_curve.

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3.0 Polynomial Expansion Of The Exponential Growth Curve

The polynomial series representation (ie, Maclaurin Series) of any infinitely differentiable function, whose value, and the values of all of its derivatives, exist at $t = 0$ is given by the infinite series

$$f(t) = f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \frac{f'''(0)}{3!}t^3 + \frac{f^{(iv)}(0)}{4!}t^4 + \frac{f^{(v)}(0)}{5!}t^5 + \dots \quad (3)$$

The Maclaurin Series of $f(t) = be^{ct}$, $t = 1, 2, 3, \dots, n$ is

$$\begin{aligned} f(t) &= be^{ct} = 1 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \dots \\ &= be^{ct} = 1 + bct + \frac{bc^2}{2!}t^2 + \frac{bc^3}{3!}t^3 + \frac{bc^4}{4!}t^4 + \frac{bc^5}{5!}t^5 + \frac{bc^6}{6!}t^6 + \dots \end{aligned} \quad (4)$$

It is clear from (4) that

$$a_k = \frac{bc^k}{k!} \quad (5)$$

In reality, exponential growth or decay does not continue indefinitely because of depletion of some rate-limiting resource. Exponential growth or decay actually refers to only the early stages of the process and to the manner and speed of growth. That is, exponential growth of physical phenomena only applies within limited region, as unbounded growth is not physically realistic.

For a time series data $X_t = f(t) = be^{ct}$, $t = 1, 2, 3, \dots, n$, this result (4) is amazing but we must worry about the range of n and c for which the polynomial approximation matches the true function curve. Outside that range of values, the polynomial approximation will be shown to be extremely inappropriate. How do we measure the approximation error in the Maclaurin's polynomial approximation to the exponential growth function? The approximation error is the discrepancy between the exact value obtained from (4) and the approximation to it obtained by fitting a polynomial of appropriate order m

$$\hat{f}(t) = \hat{a}_0 + \hat{a}_1 t + \hat{a}_2 t^2 + \dots + \hat{a}_m t^m \quad (6)$$

Table 1: True values of a_k when $-0.10 \leq r \leq 0.1$.

S/N	r	$c = \log_e(1+r)$	Maclaurin Series Values of a_k						
			a_0	a_1	a_2	a_3	a_4	a_5	a_6
1	-0.10	-0.105361	1.000000	-0.105361	0.005550	-0.000195	0.000005	0.000000	0.000000
2	-0.09	-0.094311	1.000000	-0.094311	0.004447	-0.000140	0.000003	0.000000	0.000000
3	-0.08	-0.083382	1.000000	-0.083382	0.003476	-0.000097	0.000002	0.000000	0.000000
4	-0.07	-0.072571	1.000000	-0.072571	0.002633	-0.000064	0.000001	0.000000	0.000000
5	-0.06	-0.061875	1.000000	-0.061875	0.001914	-0.000039	0.000001	0.000000	0.000000
6	-0.05	-0.051293	1.000000	-0.051293	0.001315	-0.000022	0.000000	0.000000	0.000000
7	-0.04	-0.040822	1.000000	-0.040822	0.000833	-0.000011	0.000000	0.000000	0.000000
8	-0.03	-0.030459	1.000000	-0.030459	0.000464	-0.000005	0.000000	0.000000	0.000000
9	-0.02	-0.020203	1.000000	-0.020203	0.000204	-0.000001	0.000000	0.000000	0.000000
10	-0.01	-0.010050	1.000000	-0.010050	0.000051	0.000000	0.000000	0.000000	0.000000
11	0.00	0.000000	1.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
12	0.01	0.009950	1.000000	0.009950	0.000050	0.000000	0.000000	0.000000	0.000000
13	0.02	0.019803	1.000000	0.019803	0.000196	0.000001	0.000000	0.000000	0.000000
14	0.03	0.029559	1.000000	0.029559	0.000437	0.000004	0.000000	0.000000	0.000000
15	0.04	0.039221	1.000000	0.039221	0.000769	0.000010	0.000000	0.000000	0.000000
16	0.05	0.048790	1.000000	0.048790	0.001190	0.000019	0.000000	0.000000	0.000000
17	0.06	0.058269	1.000000	0.058269	0.001698	0.000033	0.000000	0.000000	0.000000
18	0.07	0.067659	1.000000	0.067659	0.002289	0.000052	0.000001	0.000000	0.000000
19	0.08	0.076961	1.000000	0.076961	0.002961	0.000076	0.000001	0.000000	0.000000
20	0.09	0.086178	1.000000	0.086178	0.003713	0.000107	0.000002	0.000000	0.000000
21	0.10	0.095310	1.000000	0.095310	0.004542	0.000144	0.000003	0.000000	0.000000

Table 2: Regression Estimates of a_k when

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 \text{ and } -0.10 \leq r \leq 0.1; n = 100$$

S/N	r	\hat{a}_k when $R^2 = 1.00$							$ a_k - \hat{a}_k , k = 0, 1, \dots, 6$					
		\hat{a}_0	\hat{a}_1	\hat{a}_2	\hat{a}_3	\hat{a}_4	\hat{a}_5	\hat{a}_6	k = 0	k = 1	k = 2	k = 3	k = 4	k = 5
1	-0.10	0.982	-0.096	0.004	0.000	0.000	0.000	0.000	0.018	0.010	0.001	0.000	0.000	0.000
2	-0.09	0.988	-0.088	0.004	0.000	0.000	0.000	0.000	0.012	0.006	0.001	0.000	0.000	0.000
3	-0.08	0.977	-0.073	0.002	0.000	0.000	0.000	0.000	0.023	0.010	0.001	0.000	0.000	0.000
4	-0.07	0.985	-0.066	0.002	0.000	0.000	0.000	0.000	0.015	0.006	0.001	0.000	0.000	0.000
5	-0.06	0.992	-0.059	0.002	0.000	0.000	0.000	0.000	0.008	0.003	0.000	0.000	0.000	0.000
6	-0.05	0.983	-0.046	0.001	0.000	0.000	0.000	0.000	0.017	0.005	0.000	0.000	0.000	0.000
7	-0.04	0.992	-0.038	0.001	0.000	0.000	0.000	0.000	0.008	0.002	0.000	0.000	0.000	0.000
8	-0.03	0.984	-0.027	0.000	0.000	0.000	0.000	0.000	0.016	0.003	0.000	0.000	0.000	0.000
9	-0.02	0.995	-0.019	0.000	0.000	0.000	0.000	0.000	0.005	0.001	0.000	0.000	0.000	0.000
10	-0.01	0.994	-0.009	0.000	0.000	0.000	0.000	0.000	0.006	0.001	0.000	0.000	0.000	0.000
11	0.00	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
12	0.01	1.014	0.008	0.000	0.000	0.000	0.000	0.000	0.014	0.001	0.000	0.000	0.000	0.000
13	0.02	1.003	0.019	0.000	0.000	0.000	0.000	0.000	0.003	0.001	0.000	0.000	0.000	0.000
14	0.03	1.037	0.020	0.001	0.000	0.000	0.000	0.000	0.037	0.010	0.001	0.000	0.000	0.000
15	0.04	1.251	-0.026	0.005	0.000	0.000	0.000	0.000	0.251	0.066	0.004	0.000	0.000	0.000
16	0.05	2.263	-0.278	0.021	0.000	0.000	0.000	0.000	1.263	0.326	0.020	0.000	0.000	0.000
17	0.06	6.255	-1.285	0.082	-0.002	0.000	0.000	0.000	5.255	1.344	0.081	0.002	0.000	0.000
18	0.07	-5.148	2.245	-0.188	0.007	0.000	0.000	0.000	6.148	2.177	0.191	0.007	0.000	0.000
19	0.08	-22.071	8.194	-0.701	0.024	0.000	0.000	0.000	23.071	8.118	0.704	0.024	0.000	0.000
20	0.09	-79.465	27.880	-2.388	0.080	-0.001	0.000	0.000	79.465	27.794	2.392	0.080	0.001	0.000
21	0.10	96.745	-43.840	5.172	-0.248	0.006	0.000	0.000	95.745	43.935	5.168	0.248	0.006	0.000

Table 3: Regression Estimates of a_k when

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 \text{ and } -0.10 \leq r \leq 0.1; n = 50$$

S/ N	r	\hat{a}_k when $R^2 = 1.00$							$ a_k - \hat{a}_k , k = 0, 1, \dots, 6$					
		\hat{a}_0	\hat{a}_1	\hat{a}_2	\hat{a}_3	\hat{a}_4	\hat{a}_5	\hat{a}_6	k = 0	k = 1	k = 2	k = 3	k = 4	k = 5
1	-0.10	0.979	-0.093	0.004	0.000	0.000	0.000	0.000	0.021	0.012	0.002	0.000	0.000	0.000
2	-0.09	0.985	-0.086	0.003	0.000	0.000	0.000	0.000	0.015	0.008	0.001	0.000	0.000	0.000
3	-0.08	0.990	-0.078	0.003	0.000	0.000	0.000	0.000	0.010	0.005	0.001	0.000	0.000	0.000
4	-0.07	0.994	-0.069	0.002	0.000	0.000	0.000	0.000	0.006	0.003	0.000	0.000	0.000	0.000
5	-0.06	0.981	-0.055	0.001	0.000	0.000	0.000	0.000	0.019	0.007	0.001	0.000	0.000	0.000
6	-0.05	0.989	-0.047	0.001	0.000	0.000	0.000	0.000	0.011	0.004	0.000	0.000	0.000	0.000
7	-0.04	0.995	-0.039	0.001	0.000	0.000	0.000	0.000	0.005	0.002	0.000	0.000	0.000	0.000
8	-0.03	0.998	-0.028	0.000	0.000	0.000	0.000	0.000	0.002	0.001	0.000	0.000	0.000	0.000
9	-0.02	0.994	-0.019	0.000	0.000	0.000	0.000	0.000	0.006	0.001	0.000	0.000	0.000	0.000
10	-0.01	0.999	-0.010	0.000	0.000	0.000	0.000	0.000	0.006	0.001	0.000	0.000	0.000	0.000
11	0.00	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
12	0.01	1.001	0.010	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000	0.000	0.000
13	0.02	1.014	0.017	0.000	0.000	0.000	0.000	0.000	0.014	0.003	0.000	0.000	0.000	0.000
14	0.03	0.993	0.032	0.000	0.000	0.000	0.000	0.000	0.007	0.002	0.000	0.000	0.000	0.000
15	0.04	0.973	0.049	0.000	0.000	0.000	0.000	0.000	0.027	0.009	0.001	0.000	0.000	0.000
16	0.05	0.917	0.077	-0.009	0.000	0.000	0.000	0.000	0.083	0.028	0.002	0.000	0.000	0.000
17	0.06	0.785	0.130	-0.004	0.000	0.000	0.000	0.000	0.215	0.072	0.006	0.000	0.000	0.000
18	0.07	0.500	0.234	-0.010	0.000	0.000	0.000	0.000	0.500	0.166	0.013	0.000	0.000	0.000
19	0.08	1.255	-0.048	0.018	0.000	0.000	0.000	0.000	0.255	0.124	0.015	0.001	0.000	0.000
20	0.09	1.577	-0.194	0.037	-0.001	0.000	0.000	0.000	0.577	0.280	0.034	0.001	0.000	0.000
21	0.10	2.230	-0.499	0.075	-0.002	0.000	0.000	0.000	1.230	0.594	0.070	0.003	0.000	0.000

Table 4: Regression Estimates of a_k when

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 \text{ and } -0.10 \leq r \leq 0.1; n = 20$$

S/N	r	\hat{a}_k when $R^2 = 1.00$						$ a_k - \hat{a}_k , k = 0, 1, \dots, 6$					
		\hat{a}_0	\hat{a}_1	\hat{a}_2	\hat{a}_3	\hat{a}_4	\hat{a}_5	\hat{a}_6	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
1	-0.10	0.992	-0.099	0.004	0.000	0.000	0.000	0.008	0.006	0.001	0.000	0.000	0.000
2	-0.09	0.995	-0.090	0.003	0.000	0.000	0.000	0.000	0.005	0.004	0.001	0.000	0.000
3	-0.08	0.996	-0.080	0.003	0.000	0.000	0.000	0.000	0.004	0.003	0.001	0.000	0.000
4	-0.07	0.995	-0.071	0.002	0.000	0.000	0.000	0.000	0.002	0.002	0.000	0.000	0.000
5	-0.06	0.988	-0.055	0.001	0.000	0.000	0.000	0.000	0.012	0.007	0.001	0.000	0.000
6	-0.05	0.992	-0.047	0.001	0.000	0.000	0.000	0.000	0.008	0.004	0.001	0.000	0.000
7	-0.04	0.996	-0.039	0.001	0.000	0.000	0.000	0.000	0.004	0.002	0.000	0.000	0.000
8	-0.03	0.998	-0.029	0.000	0.000	0.000	0.000	0.000	0.002	0.001	0.000	0.000	0.000
9	-0.02	0.999	-0.020	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000	0.000
10	-0.01	1.000	-0.010	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
11	0.00	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
12	0.01	1.000	0.010	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
13	0.02	1.001	0.019	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000	0.000
14	0.03	1.003	0.028	0.001	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000	0.000
15	0.04	1.008	0.035	0.001	0.000	0.000	0.000	0.000	0.008	0.004	0.000	0.000	0.000
16	0.05	1.016	0.041	0.002	0.000	0.000	0.000	0.000	0.016	0.008	0.001	0.000	0.000
17	0.06	1.030	0.044	0.003	0.000	0.000	0.000	0.000	0.030	0.014	0.002	0.000	0.000
18	0.07	1.051	0.043	0.005	0.000	0.000	0.000	0.000	0.051	0.025	0.003	0.000	0.000
19	0.08	0.989	0.085	0.001	0.000	0.000	0.000	0.000	0.011	0.008	0.002	0.000	0.000
20	0.09	0.981	0.101	0.001	0.010	0.000	0.000	0.000	0.019	0.014	0.003	0.010	0.000
21	0.10	0.969	0.119	0.000	0.000	0.000	0.000	0.000	0.031	0.023	0.004	0.000	0.000

To determine m , we use the fitted polynomial equation (6) for which $R^2 = 1.00$ where R^2 is the coefficient of the multiple determination between $f(t)$ and $\hat{f}(t)$ [4]. The absolute error for each parameter for generated data of size n is given by

$$\varepsilon_k^{(n)} = |a_k - \hat{a}_k|, k = 1, 2, 3, \dots, m \quad (7)$$

Without loss of generality, we fix $b = 1$ and study the effect of c for $n = 20, 50, 100$ and $-0.10 \leq r \leq 0.10$. Table 1 gives the true values of a_k , $k = 1, 2, \dots, 6$ computed from (5). Tables 2, 3 and 4 compute the approximation errors for $n = 20, 50, 100$.

4.0 Differencing A Polynomial Of Order M

Theorem 1. If $f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots + a_r t^m$, then

$$\nabla^m f(t) = m! a_m \quad (8)$$

5.0 Differencing The Exponential Growth Curve

Theorem 2. If $f(t) = b e^{ct} = b(1+r)^t$, then

$$\nabla^d f(t) = b \left(1 - e^{-c}\right)^d e^{ct} = b \left(\frac{r}{1+r}\right)^d (1+r)^t \quad (9)$$

In particular, when $r = -0.5$,

$$\nabla^d f(t) = b(-1)^d (0.5)^t \quad (10)$$

6.0 Value Relatives

Theorem 3. If $X_t = b e^{ct} = b(1+r)^t$, $t = 1, 2, 3, \dots, n$, then

$$Y_t = \frac{X_{t+1}}{X_t} = \frac{be^{c(t+1)}}{be^{ct}} = e^c = (1+r) \quad (11)$$

7.0 Stationarity And Analysis

7.1. If the time series data, X_t , $t = 1, 2, \dots, n$, admits the exponential growth curve $M_t = be^{ct}$, $t = 1, 2, \dots, n$, we achieve stationarity and fit an autoregressive moving average process of order p and q [ARMA(p, q)]

$$Y_t = \lambda + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \dots + \phi_p Y_{t-p} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \dots - \theta_q e_{t-q} \quad (12)$$

where Y_t is the transformed series and $e_t \sim N(0, \sigma^2)$.

7.2: When process is deterministic, estimate trend and analysis is on

$$\text{Additive Model: } Y_t = X_t - \hat{b}e^{\hat{c}t}, t = 1, 2, \dots, n \quad (13)$$

$$\text{Multiplicative Model: } Y_t = \frac{X_t}{\hat{b}e^{\hat{c}t}}, t = 1, 2, \dots, n \quad (14)$$

7.3: Linearisation: Given data X_t , $t = 1, 2, \dots, n$, analysis is on the first difference of the log transformed series

$$Y_t = \log_e X_{t+1} - \log_e X_t, t = 1, 2, \dots, n_0 = n - 1 \quad (15)$$

7.4: If exponential growth curve admits adequate polynomial expansion of order m , analysis is on

$$Y_t = \nabla^m X_t, t = 1, 2, \dots, n_0 = n - m \quad (16)$$

where $\nabla^d X_t = (1 - B)^d X_t$, $B^j X_t = X_{t-j}$ is the backwards difference operator.

7.5: For all exponential growth curve models, analysis can be based on the value relatives

$$Y_t = \frac{X_{t+1}}{X_t}, t = 1, 2, \dots, n_0 = n - 1 \quad (17)$$

8.0 Illustrative Example

8.1: DATA: All Shares Index (1985 – 2007), $n = 276$

First 22 years (1985 – 2006) was used for model building and last year (2007) used for forecasting.

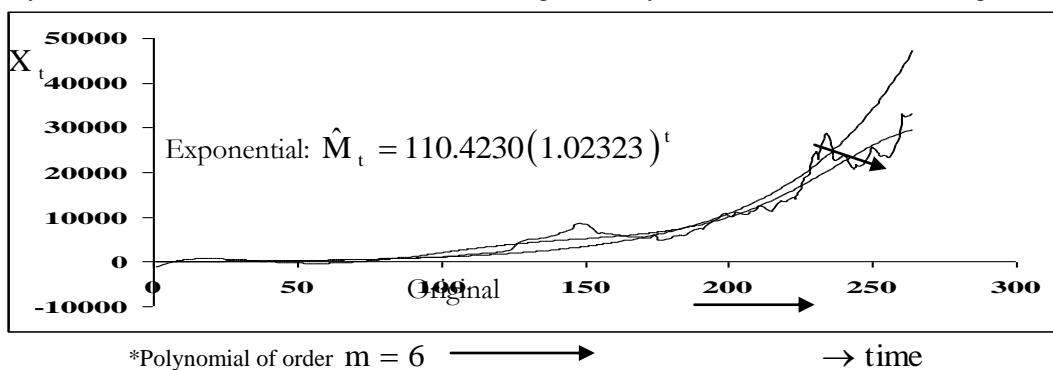


Figure 1: Plot of polynomial of order 6 and exponential growth curve

$$*\hat{M}_t = -1373.8064 - 274.6060t - 11.7834t^2 + 0.1960t^3 - 0.0015t^4$$

8.2: Assume process is deterministic

(i). Additive: $Y_t = X_t - 110.4230(1.02323)^t$, $t = 1, 2, \dots, 264$

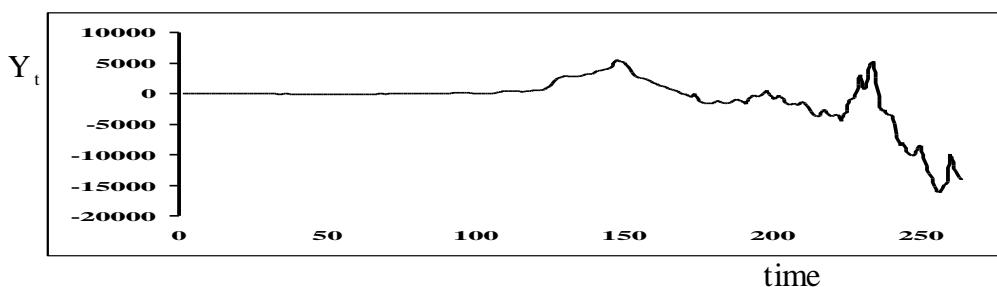


Figure 2: Plot of $Y_t = X_t - 110.4230(1.02323)^t$, $t = 1, 2, \dots, 264$

(ii). Multiplicative: $Y_t = X_t / 110.4230(1.02323)^t$, $t = 1, 2, \dots, 264$

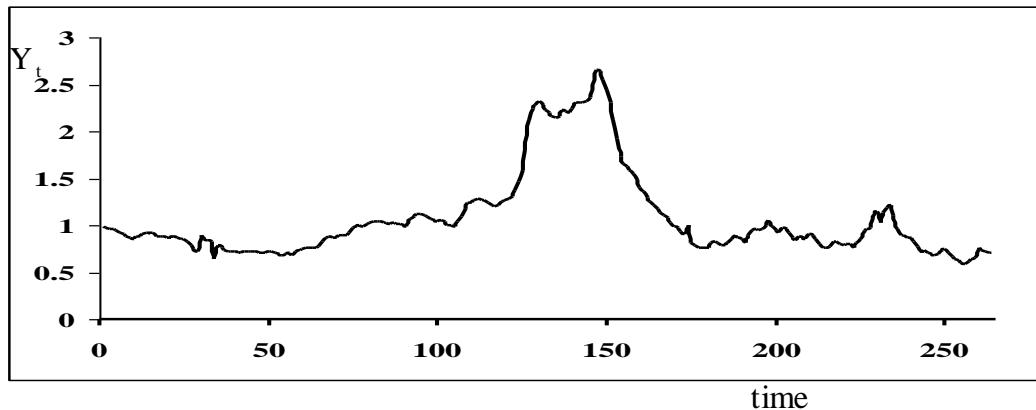


Figure 3. Plot of $Y_t = X_t / 110.4230(1.02323)^t$, $t = 1, 2, \dots, 264$

8.3: Assume process is stochastic

(i). Linearization: $Y_t = \log_e X_{t+1} - \log_e X_t$, $t = 1, 2, \dots, n_0 = n - 1 = 263$.

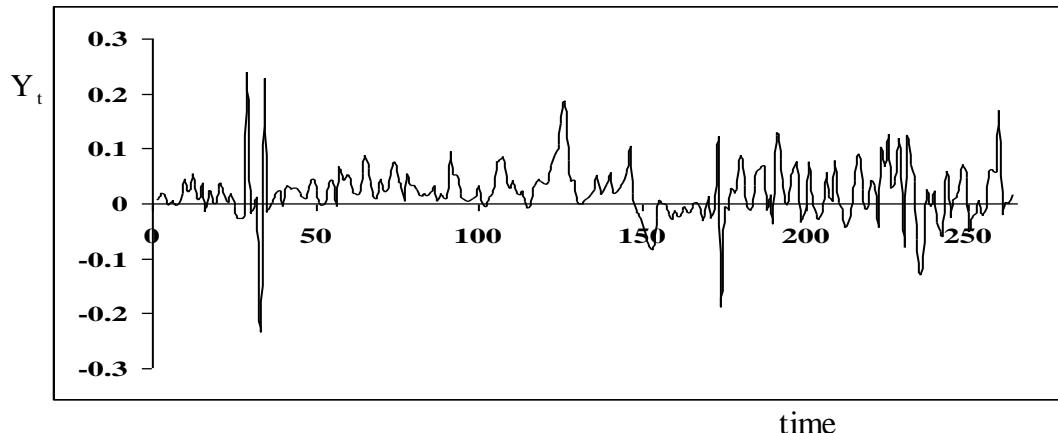


Figure 4: Time series plot of $Y_t = \log_e X_{t+1} - \log_e X_t$.

Optimal ARMA(p, q) is the zero mean ARMA(2, 3) with

$$\hat{\phi}_1 = -0.8607 \pm 0.1928$$

$$\hat{\phi}_2 = -0.4152 \pm 0.1762$$

$$\hat{\theta}_1 = -1.1200 \pm 0.1856$$

$$\hat{\theta}_2 = -0.8596 \pm 0.1701$$

$$\hat{\theta}_3 = -0.3535 \pm 0.0653, \hat{\sigma}^2 = 0.0025$$

(ii). Does adequate polynomial expansion of order r exist?.

Differencing When Growth Curve Is Exponential

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Table 5: Polynomial fits to the All Shares Index

r	\hat{a}_r	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7
0	\hat{a}_0	-5767.4100	2258.1800	-948.9389	722.5912	1173.4762	-1373.8064	-592.8192
1	\hat{a}_1	96.5500	-84.4800	59.3884	-64.6553	-114.3822	274.6060	117.6051
2	\hat{a}_2		0.6800	-0.67716	1.4256	2.7292	-11.7834	-3.9110
3	\hat{a}_3			0.0034	-0.0089	-0.0220	0.1960	0.0322
4	\hat{a}_4				0.0000	0.0001	-0.0015	0.0002
5	\hat{a}_5					0.0000	0.0000	0.0000
6	\hat{a}_6						0.0000	0.0000
7	\hat{a}_7							0.0000
R^2		75.4	93.0	94.9	95.3	95.4	96.0	96.0

Table 6: True polynomial coefficients computed from (5) using the fitted exponential growth

$$\hat{M}_t = 110.4230(1.02323)^t$$

r	0	1	2	3	4	5	6
\hat{a}_r	110.4230	2.5353	0.0291	0.0002	0.0000	0.0000	0.0000

Note that the polynomial expansion is not adequate and no amount of differencing will remove the trend.

(iii) Use Value Relatives: $Y_t = \frac{X_{t+1}}{X_t}$, $t = 1, 2, \dots, n_0 = n - 1 = 263$

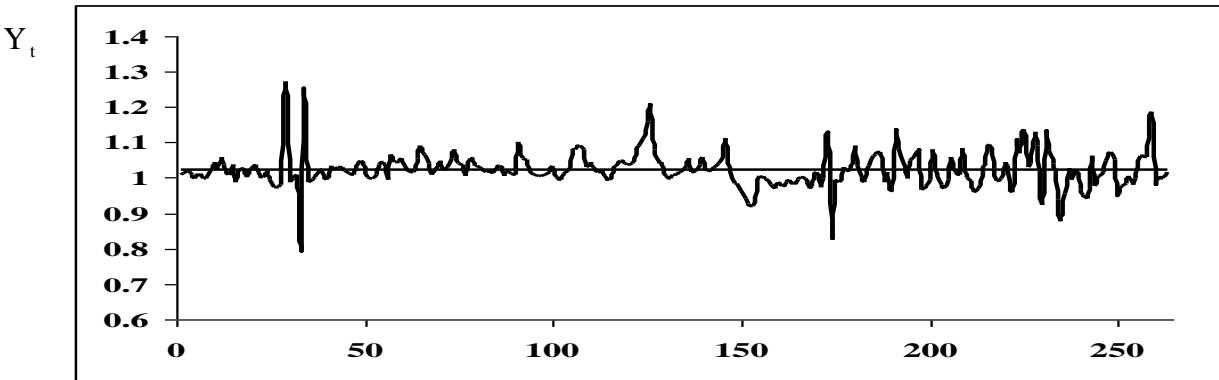


Figure 5: Time series plot of $Y_t = \frac{X_{t+1}}{X_t}$.

Optimal ARMA(p, q) is ARMA(2, 3) with

$$\hat{\lambda} = 2.6726, \hat{\phi}_1 = -1.0122 \pm 0.1633$$

$$\hat{\phi}_2 = -0.5999 \pm 0.1533$$

$$\hat{\theta}_1 = -1.1900 \pm 0.1607$$

$$\hat{\theta}_2 = -0.9408 \pm 0.1454$$

$$\hat{\theta}_3 = -0.2862 \pm 0.0628, \hat{\sigma}^2 = 0.0024$$

(iv). Comparison of ARMA(2, 3) models from $Y_t = \log_e X_{t+1} - \log_e X_t$ and $Y_t = \frac{X_{t+1}}{X_t}$

Table 7: Comparison of ARMA(2, 3) models: $Y_t = \log_e X_{t+1} - \log_e X_t$ Versus $Y_t = \frac{X_{t+1}}{X_t}$

ℓ	Original (X_t)	$Y_t = \log_e X_{t+1} - \log_e X_t$	$Y_t = \frac{X_{t+1}}{X_t}$		
		Forecast ($\hat{X}_t(\ell)$)	Error	Forecast ($\hat{X}_t(\ell)$)	Error
1	36784.5	34314.8	2469.7	34822.9	1961.6
2	40730.7	34350.7	6380.0	35393.3	5337.4
3	43456.1	34272.5	9183.6	36097.6	7538.5
4	47124.0	34324.9	12799.1	37201.1	9922.9
5	49930.2	34312.2	15618.0	37858.1	12072.1
6	51330.5	34301.3	17029.2	38779.2	12551.3
7	53021.7	34316.0	18705.7	39761.1	13260.6
8	50291.1	34307.9	15983.2	40568.6	9722.5
9	50229.0	34308.8	15920.2	41574.3	8654.7
10	50201.8	34311.4	15890.4	42541.3	7660.5
11	54189.9	34308.8	19881.1	43482.3	10707.6
12	57990.2	34309.9	23680.3	44534.2	13456.0
		MSE	241,351,090	MSE	98,991,013

9.0 Conclusion

* Trend curve of a nonseasonal time series X_t , $t = 1, 2, 3, \dots, n$ is dominated by the exponential growth curve

$$M_t = b e^{ct} \quad (18)$$

* Linearise by taking the logarithmic transform: $Z_t = \log_e X_t$. Fit ARMA(p, q) model to the first order differenced series:

$$Y_t = \nabla Z_t = \log_e X_{t+1} - \log_e X_t = c = \log_e (1 + r) \quad (19)$$

* If expected exponential curve admits adequate polynomial expansion of order m, difference series m times,

$$Y_t = \nabla^m X_t \quad (20)$$

and fit the optimal ARMA(p, q) model to Y_t .

* Always use value relatives

$$Y_t = \frac{X_{t+1}}{X_t} = e^c = 1 + r \quad (21)$$

by fitting the optimal ARMA(p, q) model to Y_t

References

- [1] Brockwell, P. J. and Davies, R. A. (1987). Time Series: Theory and methods, Springer-Verlag, New York
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- [3] Box, G. M. and Jenkins, G. M. (1976). Time Series Analysis: Forecasting and Control, 2nd Edition, Holden-Day, San Francisco.
- [4] Draper, N. R. and Smith, H. (1981). Applied Regression Analysis, Second Edition, John Wiley and Sons Inc., New York.