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Discrete Time Queueing Model for Queues at Fixed Control Signalized Intersection

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Abstract

We consider an intersection in a road network. The intersection is controlled by a fixed-cycle traffic light which alternates between red and green periods. When vehicles arrived during the red phase, they form a queue and are allowed to leave on first in, first out (FIFO) bases during the green phase. Vehicles that arrive during the green phase to meet an empty queue are allowed to proceed without waiting. We develop a discrete time queueing model for this system. Most of the previous research on queues at fixed control signalized intersection use probability generating function to derive steady state measures of performance for the system. We develop a computer program that can be used to obtain time-dependent measures of performance for the system. The model and the method of solution proposed in this paper can be used to study a broad spectrum of arrival distributions and not just the Poisson distribution alone.

1.0 Introduction

In this paper, we consider an intersection in a road network. The intersection is controlled by a fixed-cycle traffic light which alternates between red and green periods. When vehicles arrived during the red phase, they form a queue and are allowed to leave on first in, first out (FIFO) bases during the green phase. Vehicles that arrive during the green phase to meet an empty queue are allowed to proceed without waiting. We develop a discrete time queuing model for this system. Most of the previous research on queues at fixed control signalized intersection use probability generating function to derive steady state measures of performance for the system. This approach requires root finding techniques in order to invert the resulting probability generating function. Besides, only the steady state measures of performance are obtainable by the probability generating function based method. Hence, we develop a computer program that can be used to obtain time-dependent measures of performance for the system. The time dependent probabilities of the number of customers in the system can be used to derive other measures of performance.

Our model and proposed method of solution can cope with a broad spectrum of arrival distributions and not just the Poisson distribution alone. Further, the proposed model can be used to obtain insight to observed phenomenon on road traffic queues. The model is time dependent.

2.0 Literature Review

The study of queues at road intersection has received considerable attention from researchers. Hence there is a burgeoning literature on this subject. In what follows, we give a cursory review of the articles relevant to the work reported here.

According to Rouphail et al [1], "traffic delays and queues are principal performance measures that enter into the determination of intersection level of service (LOS), in the evaluation of the adequacy of lane lengths, and in the estimation of fuel consumption and emissions". They gave a detailed presentation of the evaluation of the delay and queue length models for the performance of road intersections controlled by traffic signals. They examined the merits and disadvantages of several models especially steady state queueing approach, fluid theory approach and the coordinate transformation method. Further they suggested the need for time dependent models that are applicable to the range of traffic flows when the traffic intensity is close to one.

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Kakooza et al [2] used the M/M/S queueing model to analyze the performance of unsignalized, signalized and roundabout road intersections. Using probability generating functions, they obtained expressions for the steady state expected number and waiting time of vehicles for single and double lane road links stopping at a road intersection interrupted by delays. Indeed, for a single lane system (M/M/1), Kakooza et al [2] stated that the expression for the expected number of vehicles in the system is given by

$$E(X) = \frac{\lambda[(r+f)^2 + \mu_0 f}{(r+f)[r(\mu_0 - \lambda) - \lambda f]} \tag{1.1}$$

where λ is the average number of vehicles arriving at an intersection per unit time; r is the rate of disappearance or clearance of the delays; f is the rate of occurrence of delays; and μ_0 denotes the service rate with no delays.

Based on their study, Kakooza et al [2] concluded that "under heavy traffic, signalized intersections are seen to perform better compared to un-signalized and roundabout intersections in terms of waiting time"

$$W = \frac{E(X)}{\lambda} \tag{1.2}$$

Dion et al [3] compared "the delays that are estimated by a number of existing delay models for a signalized intersection approach controlled in fixed-time and operated in a range of conditions extending from under-saturated to highly saturated. Specifically, the paper compares the delay estimates from a deterministic queuing model, a model based on shock wave theory, the steady-state Webster model, the queue-based models defined in the 1981 Australian Capacity Guide, the 1995 Canadian Capacity Guide for Signalized Intersections, and the 1994 and 1997 versions of the Highway Capacity Manual (HCM), in addition to the delays estimated from the INTEGRATION microscopic traffic simulation software." They concluded that all delay models produce similar results for signalized intersections when the traffic intensity is low. But under heavy traffic, noticeable differences occur.

Baykal-Gürsoy et al [4] used Markovian queueing theory to model the traffic flow on a road way link subject to incidents. Using M/M/S and M/M/∞ queuing systems, they obtained the steady state distribution of the number of vehicles on a road link. Their approach, method of analysis and main mathematical results are similar to those of Kakooza et al [2]. They identified excess demand for road space, irregular occurrences such as traffic accidents, vehicle disablements, spilled loads and hazardous materials as causes of congestion on the road. They also cited numerous papers on modeling traffic flow.

Viti and van Zuylen [5] examined numerous models for queues at fixed control signals. The models include deterministic models based on fluid theory, steady state queueing models, static models, heuristic approach, and time dependent queueing model based on the deterministic mass balance equation $Q(t) = \max\{Q(t-1) + t_c.a - t_g s, 0\}$. Further, Viti and van Zuylen [5] observed that the existing models do not provide "insight into the way queues are experienced by drivers at signalized intersection". They proposed a probabilistic model for queues at fixed control signals. Their work is closest to the one reported in this paper. Later, we shall identify the differences between their work and the one reported here.

3.0 Model Development

3.1 Notation

$A1(k)$ the probability of k arrivals during the red phase.

$A2(k)$ the probability of k arrivals during the green phase.

$PQR(j)$ the probability that there are *j* cars in the queue at the end of the red phase.

$PQG(j)$ the probability that there are *j* cars in the queue at the end of the green phase.

$P1(i)$ the probability of *i* cars in the queue at the beginning of the red phase,

$P2(i)$ the probability of *i* cars in the queue at the beginning of the green.

s the number of departures during the green phase.

λ arrival rate for Poisson arrival process

Q_{max} Maximum queue length

3.2 Assumptions

We shall assume that the arrivals occur singly and are assumed independently distributed. For illustrative purpose, we shall use Poisson arrival process with parameter λ. The service time during the green phase is s a constant, s stands for the maximum number of vehicles that can leave the queue during the green phase. There is only a single land and overtaking is prohibited. Once the traffic light is green, a maximum of s vehicles can escape to infinity. This means that there is no queue ahead of the vehicles using the traffic light. We also consider vehicles as homogeneous. This means that the number of vehicles that can benefit when the traffic light is green is the same all the time.

In order, to incorporate divers arrival processes, we shall examine the system at discrete time epoch. The main epochs are (a) the beginning and end of a red phase (b) the beginning and end of a green phase. These epochs are called regeneration points; see Cox and Smith [6]. We examine the system at discrete time epochs, namely at the end of the red phase and green phase. The amber phase is ignored. We assume that the amber phase is either part of the red phase or part of the green phase because in practice, the end of a red phase is the beginning of a green phase and the end of a green phase is the beginning of a red phase.

3.3 The red phase

The red phase is from the moment the red light comes on to when it is off. Let i be the number of cars in the queue at the beginning of the red phase. During the red phase, transition from state i to state j is impossible if $i > j$ since there are no departures during the red phase. If $i \le j,$ the transition probability from state i to state j is the probability of $k = j - i$ arrivals during the red phase. This probability we shall denote by $A1(k).$ Therefore, if $P1(i)$ is the probability of i cars in the queue at the beginning of the red phase, then $PQR(j)$ the probability that there are j cars in the queue at the end of the red phase is given by

$$PQR(j) = \sum_{i=0}^{j} P1(i)A1(j-i), \qquad j = i, i+1, \ldots, Q_{\max} \tag{3.1}$$

and $PQR(j) = 0$ otherwise.

3.4 The green phase

Suppose that there are i cars in the queue at the beginning of a green phase. Let $P2(i)$ be the probability of i cars in the queue at the beginning of the green. During this phase, let k be the number of arrivals with probability $A2(k).$ If s is the number of departures, then j the number of cars in the queue at the end of the green phase is given by $j = \max\{i + k - s, 0\}$, see Cox and Smith [6]. The transition from state i to state 0 during the green phase is a special case and shall be treated separately.

First, observe that if $i > s,$ transition from state i to state 0 is impossible. Second, if $i \le s,$ transition from state i to state 0 occurs if the number of arrivals is $k \le s - i.$ This shows that Equation (11) of Viti and van Zuylen [5] is miss leading. We enumerate the elementary events that can give rise to state zere (0) for all $i \le s,$ i.e. $i = 0, 1, \ldots, s.$ Table 1 shows the decomposition of the events leading to state zero at the end of a green phase starting from state $i = 0, 1, \ldots, s$ at the beginning of a green phase. If $i = 0,$ transition to state 0 can occur if we have $0, 1, \ldots,$ or $s,$ arrivals. If $i = 1,$ transition to state zero is possible if we have $i = 0, 1, \ldots, s-1,$ arrivals and so on, until if $i = s,$ transition to state zero is possible if we have no arrival.

Consequently, the probability of having no cars in the queue at the end of the green phase is given by

Table 1. Decomposition of the events

Case	Number in the queue at the beginning of green phase	Elementary events	Probability
1	0	0 or 1 or 2 … or s arrivals	$P2(0)\sum_{i=0}^{s} A2(i)$
2	1	0 or 1 or 2 … or $s-1$ arrivals	$P2(0)\sum_{i=0}^{s-1} A2(i)$
⋮	⋮	⋮	⋮
s	$s-1$	0 or1 arrival	$P2(s-1)\sum_{i=0}^{1} A2(i)$
$s+1$	s	0 arrival	$P2(s)A2(0)$

$$PQG(0) = \sum_{k=o}^{s} P2(k)(\sum_{i=0}^{s-k} A2(i)) \tag{3.2}$$

The probability of having $j > 0,$ cars in the queue at the end of the green phase is obtained as follows. First observe that if $j + s - i < 0,$ transition from state *i* to state *j* is impossible since arrivals are constrained to be non-negative. So for transition from state *i* to state *j* to occur, we must have $j + s - i \geq 0,$ and the probability of $j > 0,$ cars in the queue at the end of the green phase is given by

$$PQG(j) = \sum_{i=0}^{j+s} P2(i)A2(j+s-i), \quad j = 1, 2, \ldots \tag{3.3}$$

Equations (3.1), (3.2) and (3.3) give a complete specification for the probability of having *j* cars in the queue at the end of the regeneration points of the system. Equation (3.2) reported here is different from that given by Viti and van Zuylen [5] for the calculation of the probability that the system is empty during the green phase. According to Viti and van Zuylen [5], "Since queues are constrained to be non-negative, when the possible departures are larger than the sum of arrivals and queue at the starting of the cycle the queue at the end of the green phase will be zero. The chance of a queue i to become zero is therefore computed with the following condition:"

$$P_{i0} = \sum_{n=0}^{i-s} P_a(n < i - s) \quad \forall i - s \leq a_{\max}$$

where $P_a(n)$ is the probability of the arrivals being a value n, and $a_{\max}$ is the maximum arrival rate. We have given the correct version of this probability in our equation (3.2). In the next section we describe how to obtain numerical values for theses probabilities. As argued by Viti and van Zuylen [5], other important measures of performance of the signalized intersection can be obtained from these distributions.

4.0 Solution

One method of solving equations (3.1), (3.2), and (3.3) is to assume steady state conditions and then use probability generating function, see Kakooza et al [2] and Baykal- Gürsoy et al [4] for examples of this procedure. But this approach requires the roots of the generating function to obtain the probabilities. This is usually a formidable task and people resort to obtaining only the expected number of cars in the queue by differentiating the generating function as demonstrated by Kakooza et al [2] and Baykal-Gürsoy et al [4]. Even, for simple cases, it is extremely difficult to use the method of generating function to obtain transient results, see Cox and Smith [6]. Cox and Smith [6] considered the single server queue with random arrival and exponential service times. They obtained the non –equilibrium probability generating function for the probability that there are *n* customers in the queue at time *t,* including the one being served. Cox and Smith [6] concluded that the solution obtained is "far from convenient; when we consider that it originates from one of the simplest queueing systems". In our case, the arrival process can be non-stationary.

Koopman [7] considered a problem similar to our own where the arrival process is time dependent. He used computer program to solve the equations for the probabilities and discussed the problem of choosing an upper limit to the queue length. The choice of an upper limit to the queue length is indeed a practical issue since in practice, the queue length is finite. Hence we shall develop computer programs to generate the time dependent probabilities of having *j* cars in the queue at the end of the regeneration points of the system. The program has a very simple structure. The results reported in this paper were obtained using a program written in MATLAB.

We can begin with either the red phase or the green phase. Figure 1 shows the graph of the expected queue length when λ=0.1, s=1 and the initial number of vehicles in the queue is 5. The system started from the red phase.

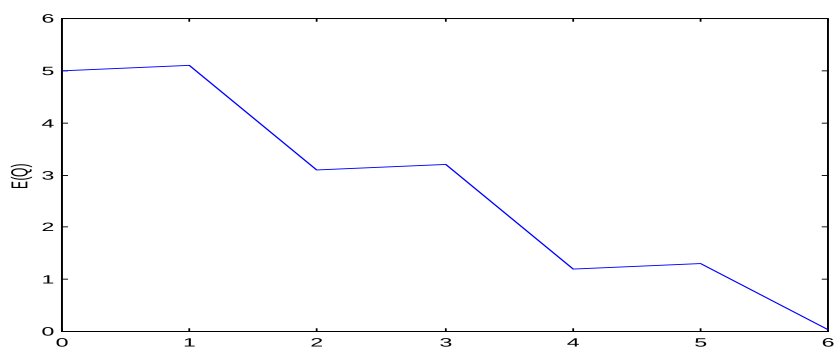

Figure 1: The expected queue length when λ=0.1, s=1 and the initial number of vehicles in the queue is 5. The odd numbers in the time axis stand for end of red phase and the even numbers stand for end of green phase.

Figure 2 shows the graph of the expected queue length when λ=0.1, s=1 and the initial number of vehicles in the queue is 5. The system started from the green phase. Figure 3 shows the expected queue length when λ=0.1, s=1 and the initial number of vehicles in the queue is 0. The odd numbers in the time axis stand for end of red phase and the even numbers stand for end of green phase.

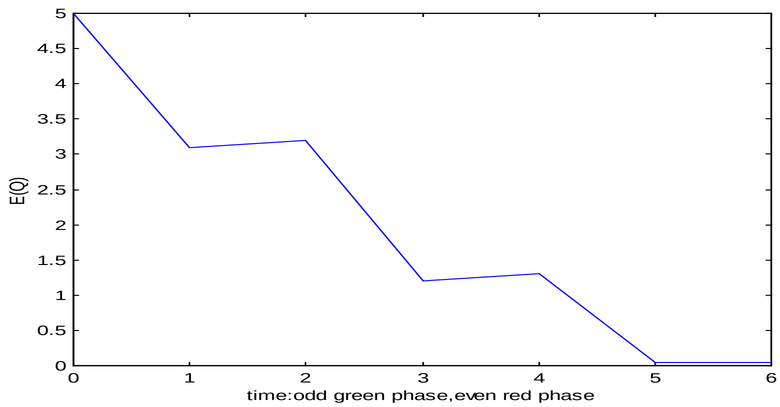

Figure 2: The expected queue length when λ=0.1, s=1 and the initial number of vehicles in the queue is 5. The odd numbers in the time axis stand for end of green phase and the even numbers stand for end of red phase.

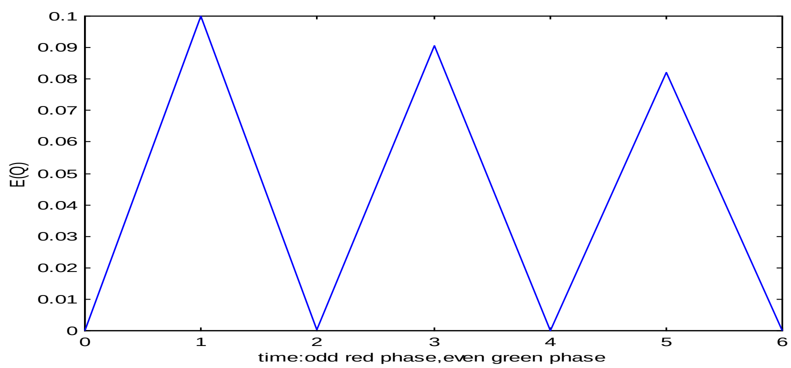

Figure 3: The expected queue length when λ=0.1, s=1 and the initial number of vehicles in the queue is 0. The odd numbers in the time axis stand for end of red phase and the even numbers stand for end of green phase.

The saw-tooth shape shown in Figure 3 is what is expected as empirical observation shows that there is a tendency for the queue length to increase during the red phase and decrease during the green phase when the traffic intensity is less than one. Observe the difference between the case when the system starts with the red phase and when it starts with the green phase (Figures 1 and 2). Also observe the difference between the case when the system starts empty and when the queue length is greater than zero initially (compare Figure 1and Figure 3).

Clearly, the model proposed in this paper is able to describe the observed behavior of real signalized intersection. With the model describe in this paper, queues at signalized intersection with non-stationary arrival process can be examined. What is required is to change the matrix representing the arrival process whenever it is necessary to do so.

5.0 Discussions

In the forgoing, it was assumed that the vehicles were homogeneous and that the drivers behaved in a uniform manner. Hence the number of vehicles that can leave the system during a green phase is s, a constant. In practice these two assumptions are unrealistic. One way of accounting for this is to do an empirical study and associate with the departure process an empirical distribution. Another option is to do a sensitivity analysis. The work reported here can easily be incorporated into a multi-scale model of road traffic flow. The method of writing computer program to provide the desired probabilities for discrete time queueing models is well known, see for example Omosigho and Worthington [8]. The approach is very flexible and can accommodate different arrival patterns including platoon arrival process. In queueing theory, platoon arrival is similar to bulk arrival.

6.0 Conclusion

The results presented in this paper show the flexibility of the proposed model. Previous errors in the work of viti and van Zulen have been corrected. A number of models have been presented in the past for the analysis of signalized intersection. Majority of them provides steady state measures. But our model can produce transient results and can permit changes in the arrival process. Changes in the service process can also be accommodated. The model can be improved in several directions. First, multiple lanes can be considered. Second, the number of vehicles permitted to use the signalized intersection during the green phase can be a random variable.

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