Journal of the Nigerian Association of Mathematical Physics
***Volume* 22 (November, 2012), pp 43 – 48**

The Ising Model on the Bethe Lattice

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Abstract

The Ising model is a simple model that can be solved exactly on the Bethe lattice. In this work, a review is undertaken highlighting the very important fact that the Ising model on the Bethe lattice is equivalent to an approximate treatment of the Ising model on crystalline lattices and site percolation in the Bethe Lattice. It is observed that the results for the Bethe lattice are typically below the crystalline results by 5-15%. However, the Bethe lattice results have the correct trend of decrease as the branching ratio increases.

Keywords: Bethe lattice, crystalline lattices, cayley tree and Ising model

1.0 Introduction

The study of exactly solved models has elicited some general interest in statistical mechanics. The statistical model on the Bethe lattice of general connectivity, with hopping interactions between nearest neighbor (NN) and next nearest neighbor(NNN) spin is one of the models that has received widespread attention by many authors recently [1-6]. The Bethe lattice is an infinite graph where any two parts are connected by a single path and each vertex has the same number of branches *z*. Each site has *z* neighbors, but there is only one way to get from one site to another. Following the mean field theory of magnetism in which an effective field is placed on a single site, the Bethe-Peierls approximation was introduced to describe crystalline alloys or Ising models. It was pointed out later [7] that this approximation becomes exact on the Bethe lattice. A Bethe lattice, which is an infinitely Cayley tree (a finite portion of the Bethe lattice) is a connected graph without circuits and historically gets its name from the fact that its partition function is exactly that of an Ising model on the Bethe approximation. The Cayley tree has a peculiar thermodynamic limit due to its large surface whereas the infinite Bethe lattice has no surface, all its lattice sites being located inside the infinite tree. In the Bethe lattice, one is concerned with properties that are unaffected by the surface although this cannot be avoided in all cases. Hence, the study of the different models on the Bethe lattice is numerically feasible [8]. The precise difference between Cayley tree and Bethe lattice is given by Baxter [9].

The Bethe lattice or the infinite Cayley tree presents a hierarchical structure that greatly simplifies some problems of statistical physics. It has therefore been widely used to obtain analytical results for problems that are otherwise intractable on Euclidean lattices. The physical relevance of these results is that the Bethe lattice is supposed to represent some mean field limit of Euclidean lattices of very large dimensions [10].

The Bethe lattice is a pseudo-lattice because it does not possess the usual point symmetries and translational symmetries of crystal Bravais lattice [11]. Nevertheless, it plays an important role in statistical and condensed matter physics because some problems involving disorder and/or interactions can be solved exactly when defined on a Bethe lattice due to its recursive structure e.g. Ising models, percolation or Anderson localization. Such exact solution on the Bethe lattice for $z < \infty$ sometimes, but not always, have mean field character. In any statistical mechanical system each component interacts with the external field and with the neighboring components. In the mean-field model, the second effect is replaced by an average over all components. Furthermore, it was argued that mean field theories are more reliable if derived on a Bethe lattice [12].

The Bethe lattice continues to be a popular model. The earlier applications included localization, alloys, spin waves and spin glasses. Recently it has been used to investigate properties of the Potts model, the Blume-Capel model, the Hubbard model and the Anderson model [2, 4]. Bethe lattice may actually serve as a model for the electronics structure of amorphous solids [13]; see [14] for more applications. The local Green function for a quantum-mechanical particle with hopping between NN and NNN on the Bethe lattice was calculated [11], where the on-site energies may alternate on sub-lattices. For infinite connectivity the renormalized perturbation expansion was carried out by counting all non-self intersecting paths, leading to an implicit equation for the local Green function. By integrating out branches of the Bethe lattice, the same equation is obtained from a path integral approach for the partition function.

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There are two special properties that make the Bethe lattice particularly suited for theoretical investigations. First is its self similar structure which may lead to recursive solutions. The other is the absence of closed loops which restrict interference effect of quantum mechanical particle in the case of NN coupling. The situation is different if also long range hopping processes or interactions are allowed say between NNN, where a frustration introduced by the NNN hopping typically suppresses anti-ferromagnetism in the half–filled Hubbard model at weak coupling. It was then considered for cases where the hopping has both NN and NNN contributions and to the limit $z \to \infty$ [12]. In practice, there are two approaches to solving the Bethe lattice. One deals with simple Hamiltonians whose interactions are restricted to first or second-nearest neighbor atoms in order to carry out the calculations analytically while in the other, more involved Hamiltonians are used and consequently the solution can only be obtained numerically [15]. In this work, the collective states of the Ising model on the Bethe lattice are investigated and its trend of increase is examined.

1. Ising Model on the Bethe Lattice

Ising type models with spin greater then ½ have rich fixed point structures. The greatest interest in these models arises partly from the unusually rich phase transition behavior they display as their interaction parameters are varied and partly from their many possible applications [16]. For the case of spin one half particles on each lattice site, the z component of spin can point either up or down, which is usually denoted by $\sigma = \pm 1$. The spin on neighboring sites $\langle ij \rangle$ then interacts by $-J\sigma_i \sigma_j$. The influence of the crystal field on the phase diagrams of the bilayer spin-1 Ising model on the Bethe lattice in terms of the interlayer coupling constants J_1 and J_2 of the two layers and interlayer coupling constant J_3 between the layers for given values of the coordination number z has been studied [17] by using the recursion relation scheme.

The Ising model with a magnetic field can be solved exactly, analytically, for a one dimensional chain. In two dimensions, it can be solved exactly without a magnetic field as first done by Onsager [18]. The Hamiltonian to solve exactly for the Bethe lattice is [19]

$$H = -J.\sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_j \qquad (1)$$

where $h \equiv \mu H$ is the Zeeman energy in a magnetic field and σ_i is a variable which takes the values from the set $\{-\sigma, -\sigma+1, \cdots \sigma-1, \sigma\}$. The first sum in (1) goes over all nearest neighbor (NN) pairs of the Bethe lattice, and the second over all sites. $J\sigma_i \sigma_j$ contains all possible nearest neighbor (NN) pair interaction and $h\sigma_j$ includes all possible single interactions. The parameter z is the coordinator number. It is the number of first neighbors for each site. A related parameter is r = z – 1, called the branching ratio. When a particle comes to a site, it has r choices of paths to move forward. Here, the particles are not moving but only their spins interact.

If we construct an Ising model on the complete Cayley tree, then the partition function Z contains contributions from both sites deep within the graph, and sites close to or on the boundary. The contribution from the latter is not negligible, even in the thermodynamic limit. If one considers the total partition function, then one is considering the 'Ising model on the Cayley tree'. This problem has been solved [20] and has some quite unusual properties and will not be considered here. Instead the contribution to Z from sites deep within the graph shall be considered, i.e. from the Bethe lattice.

If one makes a low temperature expansion for any regular lattice, then to second order the only properties of the lattice that one need to know are the number of sites and the coordination number. To third order one needs the number of triangles in the lattice, to fourth order the number of tetrahedra (i.e. clusters of 4 sites all connected to one another) and other highly connected 4-point sub-graphs, and so on. An interesting simple case is when there are no circuits at all, and hence there are no triangles, tetrahedra, etc. Then one obtains the Ising model on the Bethe lattice.

The Bethe lattice shown in Fig.1 is convenient for solving the Ising model. The bottom of the figure is called the boundary of the lattice. The partition function is obtained by averaging the spins at the boundary and then moving inward row by row.

The first step is to average all the spins in the $j = 0$ row. Each spin in the $j = 1$ row is connected to r spins in the $j = 0$ row. Let σ denote a spin on the $j = 1$ row and let $(\sigma_1, \sigma_2 \cdots \sigma_r)$ denote the r spins connected to it in the $j = 0$ row. Averaging this small contribution of the partition function Z gives

$$Z = \sum_{\sigma} \exp \beta \left\{ \sum J\sigma_i \sigma_j + \sum_j h\sigma_j \right\} \qquad (2)$$

$$\text{That is, } Z = \sum \exp \{\beta J\sigma(\sigma_1 + \sigma_2 + \cdots + \sigma_r) + \beta h(\sigma + \sigma_1 + \sigma_2 \ldots + \sigma_r)\}$$

$$= e^{\beta h \sigma} \left[2\, Cosh(\beta J\sigma + \beta h)\right]^r \qquad (3)$$

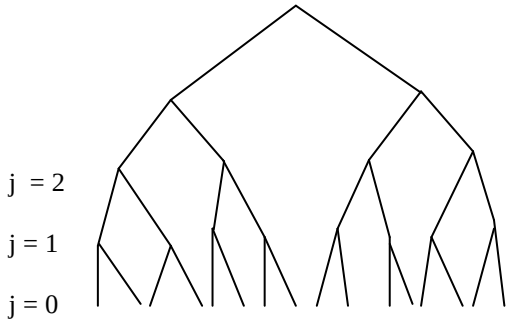

Fig. 1. Bethe Lattice for z = 3 with a boundary.

Each spin has an effective interaction with its r neighbors in the lower row of $\sigma\left[h + J\left(\sigma_1 + \cdots + \sigma_r\right)\right]$. In the nonmagnetic state, the sum over the neighbors will average out to a small number. In the magnetic state, the sum over the neighbors could add up to $\pm r$. It is useful to define the effective magnetic energy h_j which acts upon the spin in row j.

Let $B_j = \beta h_j$ (a shorthand notation), hence $h_j = \dfrac{B_j}{\beta}$

For the first row, there are no spins below, so that $h_o = h$, $B_o = \beta h$. For the second row, B_1 is defined from eqn. (3) as

$$A_1\, e^{B_1\sigma} = e^{\beta h\sigma}\left[2\, Cosh\left(B_o + \beta J\sigma\right)\right]^r \tag{4}$$

Setting $\sigma = \pm 1$ gives two equations which are solved for two unknowns (A_1, B_1):

$$A_1\, e^{B_1} = e^{\beta h}\left[2\, Cosh\left(B_o + \beta J\right)\right]^r \tag{5}$$

$$A_1\, e^{-B_1} = e^{-\beta h}\left[2\, Cosh\left(B_o - \beta J\right)\right]^r \tag{6}$$

$$B_1 = \beta h + \frac{r}{2}\ \ln\left[\frac{Cosh\left(B_o + \beta J\right)}{Cosh\left(B_o - \beta J\right)}\right] \tag{7}$$

where βh is from the energy of the spin in row $j = 1$

$$A_1 = 2^r\left[Cosh\left(B_o + \beta J\right)\ Cosh\left(B_o - \beta J\right)\right]^{1/2} \tag{8}$$

The factor of B_o is inserted instead of βh in the argument of the hyperbolic cosines since the effective field is associated with the spins on the lower row. Expanding the term inside the brackets in eqn. (7) from trigonometry, gives

$$\frac{Cosh\left(B_o + \beta J\right)}{Cosh\left(B_o - \beta J\right)} = \frac{Cosh\left(B_o\right)Cosh\left(\beta J\right) + Sin\ h\left(B_o\right)Sin\left(\beta J\right)}{Cosh\left(B_o\right)Cosh\left(\beta J\right) - Sin\ h\left(B_o\right)Sin\left(\beta J\right)}$$

dividing both the numerator and denominator by $Cosh\left(\beta_o\right)Cosh\left(\beta J\right)$ gives

$$\frac{Cosh\left(B_o + \beta J\right)}{Cosh\left(B_o - \beta J\right)} = \frac{1 + \tan h\left(B_o\right)\tan h\left(\beta J\right)}{1 - \tan h\left(B_o\ \ \tan h\ \ \beta h\right)}$$

Let $\tan h\left(B_o\right)\tan h\left(\beta J\right) = x$, then

$$\frac{Cosh\left(B_o + \beta J\right)}{Cosh\left(B_o - \beta J\right)} = \frac{1 + x}{1 - x}$$

But

$$\ln\left[\frac{1 + x}{1 - x}\right] = 2\tan h^{-1}\ x \tag{9}$$

Therefore

$$\ln\left[\frac{Cosh\left(B_o + \beta J\right)}{Cosh\left(B_o - \beta J\right)}\right] = 2\tan h^{-1}\ x$$

and substituting x gives

$$\ln\left[\frac{Cosh\left(B_o+\beta J\right)}{Cosh\left(B_o-\beta J\right)}\right] = 2\tan h^{-1}\left[\tan h\left(B_o\right)\tan h\left(\beta h\right)\right] \qquad (10)$$

Substituting (10) into equation (7) gives

$$B_1 = \beta h + r\tan h^{-1}\left[\tanh\left(B_o\right)\tanh\left(\beta h\right)\right] \qquad (11)$$

The second step is to average the spins in the row j = 1 to give the effective field $B_2 = \beta h_2$ in the next row. This will give;

For $j=1$, $B_2 = \beta h_2$

$$B_2 = \beta h + r\tan h^{-1}\left[\tanh\left(B_1\right)\tanh\left(\beta J\right)\right] \qquad (12)$$

For $j=2$, $B_3 = \beta h_3$

$$B_3 = \beta h + r\tan h^{-1}\left[\tanh\left(B_2\right)\tanh\left(\beta J\right)\right] \qquad (13)$$

Thus, the general recursion relation as the rows are averaged one by one is

$$B_{j+1} = \beta h + r\tan h^{-1}\left[\tan h\left(B_j\right)\tanh\left(\beta J\right)\right] \qquad (14)$$

Hence

$$h_j = h + \frac{1}{\beta} r\tan h^{-1}\left[\tan h\left(B_{j-1}\right)\tanh\left(\beta J\right)\right] \qquad (15)$$

As the spins are averaged row by row, the effective field $B_j = \beta h_j$ converges to the value in the interior of the Bethe lattice. This bulk value is denoted as B^*. It obeys the self consistent non-linear equation.

$$B^* = \beta h + r\tan h^{-1}\left[\tan h\left(B^*\right)\tan h\left(\beta J\right)\right] \qquad (16)$$

The solution to eqn. (16) describes the collective states of the Ising model on the Bethe lattice.

1. States of the Ising Model on the Bethe Lattice

The collective states of the Ising model on the Bethe lattice can be described from the solution to eqn. (16).

Assume the zero magnetic field (h = 0) and ferromagnetic coupling (J>0). The transition temperature T_c, $\left(\beta_c = 1/k_B T_c\right)$ is where the ordering begins as one lowers the temperature. At the transition temperature, the order parameter B^* is zero and it increases in value as the temperature is lowered. It is infinitesimally smaller than T_c, hence equation (16) is

$$B^* = r\,B^*\tan h\left(\beta_c J\right) \qquad (17)$$

But

$$\tanh\left(\beta_c J\right) = \frac{1}{r} \qquad (18)$$

hence

$$\beta_c J = \tan h^{-1}\frac{1}{r} \qquad (19)$$

And

$$\beta_c = 1/k_B T_c$$

therefore

$$\frac{J}{K_B T_c} = \tan h^{-1}\frac{1}{r} \qquad (20)$$

From eqn. (9)

$$\ln\left[\frac{1+x}{1-x}\right] = 2\tan h^{-1} x$$

Let $x = \frac{1}{r}$ then

$$\ln\left[\frac{1+\frac{1}{r}}{1-\frac{1}{r}}\right] = 2\tan h^{-1} x \qquad (21)$$

$$\tan h\,\frac{1}{2}\,\ln\left(\frac{r+1}{r-1}\right)=\frac{1}{r} \tag{22}$$

Thus

$$k_B T_c = \frac{2J}{\ln\left(\frac{r+1}{r-1}\right)} \tag{23}$$

$$K_c = \beta_c J = \frac{1}{2}\,\ln\left(\frac{r+1}{r-1}\right) \tag{24}$$

For r = 1 which is a one dimensional chain, T_c = 0 and there is no ordered state at non zero temperature. For all other branching ratios r > 1, there is a well defined transition temperature. This is shown in Table 1 for some crystals.

Table 1. Ferromagnetic transition temperature $K_c(B)$ of Ising model on Bethe lattice from eqn. (24) (d is the dimension, r is the branching ratio).

Crystal	d	r	$K_c(B)$
Honeycomb (hc)	2	2	0.549
Square (sq)	2	3	0.347
Plane triangular (pt)	2	5	0.203
Simple cubic (sc)	3	5	0.203
Body centered cubic (bcc)	3	7	0.144
Face centered cubic (fcc)	3	11	0.091

Comparing the Bethe results in Table 1 with the exact results as shown for Ising model on crystalline lattices in two and three dimensions in ref. [21] and the percolation in the Bethe lattice for some crystals given in ref [22], it can be observed that the value of $K_c(B)$ for the Bethe lattice is typically below the crystalline results by 5–15%. However, the Bethe lattice has the correct trend that K_c decreases and T_c increases as the branching ratio r increases as shown in Table 2 and Table 3.

Table 2. The ferromagnetic transition temperature of Ising model on crystalline lattices K_c and the Bethe results $K_c(B)$ (where d is the dimension and r is the branching ratio).

d	Crystal	r	K_c	$K_c(B)$
2	Honeycomb	2	0.657	0.549
2	Square	3	0.441	0.347
2	Plane triangular	5	0.274	0.203
3	Diamond	3	0.370	0.347
3	Simple cubic	5	0.222	0.203
3	Body centre cubic	7	0.157	0.144
3	Face centre cubic	11	0.102	0.091

Table 3. The site percolation and the ferromagnetic transition temperature $K_c(B)$ of Ising model on the Bethe lattice.

d	Crystals	r	$K_c(B)$	Site percolation
2	Honeycomb	2	0.549	0.696
2	Square	3	0.347	0.592
2	Plane triangular	5	0.203	0.500
3	Simple cubic	5	0.203	0.312
3	Diamond	3	0.347	0.431
3	Body centre cubic	7	0.144	0.246
3	Face centre cubic	11	0.091	0.198

2. Conclusion

The Ising model on the Bethe lattice is examined with arbitrary branching ratio r and with hopping interactions between NN, NNN spins belonging to the same branch and under an external field. The Ising model on the Bethe lattice of some crystalline lattices were calculated by solving the state of the recursion equations of appropriate effective fields numerically

and were compared to the exact results on crystalline lattices and results of the percolation in the Bethe lattices. It was clear that the Ising model on the Bethe lattice approximated closely the exact results with correct trend of decrease as the branching ratio increases.

3. References

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